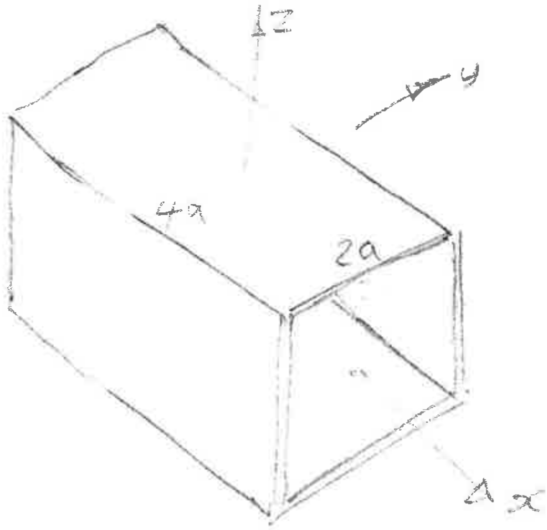
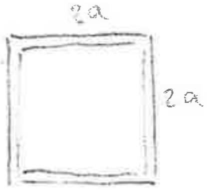
1 a/



View x



$$I_{xx} = 4\,\frac{m}{4}\left(\frac{1}{3}a^2 + a^2\right)$$

$$= \underline{\underline{\frac{4}{3}ma^2}}$$

View y

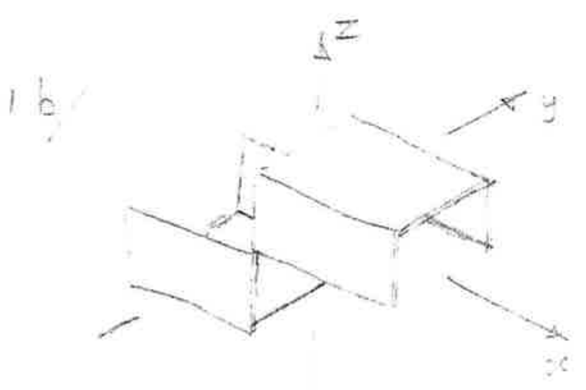
View z

$$I_{yy} = I_{zz} = 2\,\frac{m}{4}\left(\frac{1}{3}(2a)^2 + a^2\right) + 2\,\frac{m}{4}\left(\frac{1}{3}a^2 + \frac{1}{3}(2a)^2\right)$$

$$= \frac{m}{2}a^2\left(\frac{7}{3} + \frac{5}{3}\right)$$

$$= \underline{\underline{2ma^2}}$$

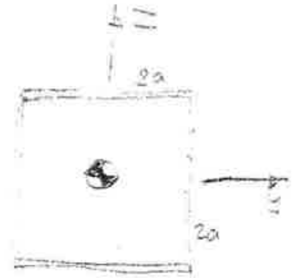
1 b)



Acceptable to argue (carefully) from symmetry in each view that the moments of inertia are all ½ the values from (a) because the mass is halved but the distribution still essentially the same.
Otherwise, details as set out below:

VIEW X

$\frac{M}{2}$



$$I_{xx} = 4\,\frac{M}{8}\left(\frac{1}{2}(2a)^2 + a^2\right)$$

$$= \frac{M}{2}\,\frac{4}{3}\,a^2$$

$$= \frac{2}{3}Ma^2$$

$I_{yz} = 0$

VIEW Y

$\frac{M}{4}$

$$I_{yy} = \frac{M}{4}\cdot\frac{1}{3}\left(a^2 + \ldots\right)$$

$$+ \ldots\,\frac{1}{3}\,(2a)^2 + \ldots\,a^2$$

$$= \ldots = Ma^2$$

For I_{xy}, use plates + parallel axis, or: $I_{xz} = \int xz\,dm$

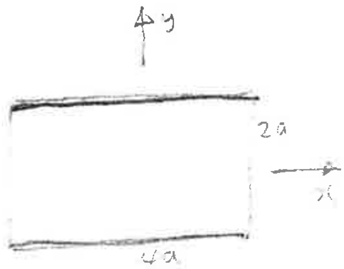
$I_{xz_1} = 2\int_0^{2a}$

$I_{xz} = \ldots$

View Z

plate $\frac{m}{4}$

rods $\frac{m}{4}$



$$I_{zz} = \frac{m}{4}\frac{1}{3}\left(a^2 + (2a)^2\right) + \frac{m}{4}\left[\frac{1}{3}(2a)^2 + a^2\right]$$

$$= \frac{ma^2}{4}\left(\frac{5}{3} + \frac{7}{3}\right)$$

$$= ma^2 \quad (= I_{yy} \checkmark)$$

$$I_{xy} = 0$$

$$\therefore \quad I = ma^2 \begin{bmatrix} \frac{2}{3} & 0 & -\frac{3}{8} \\ 0 & 1 & 0 \\ -\frac{3}{8} & 0 & 1 \end{bmatrix}$$

1c/ $\quad \underline{Q} = \underline{\dot{h}} = \frac{d}{dt}\left(I\,\underline{\omega}\right)$

$$\underline{\omega} = \Omega\,\underline{i} = \text{const}$$

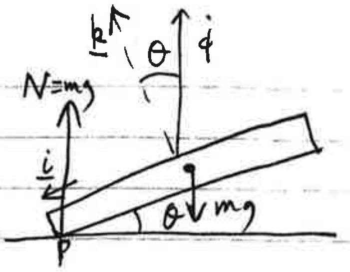
$$\therefore \underline{Q} = \underline{\omega} \times \underline{h} = \underline{\omega} \times (I\,\underline{\omega})$$

$$= \Omega\,\underline{i} \times \left(\frac{2}{3}\Omega\,\underline{i} - \frac{3}{8}\Omega\,\underline{k}\right) ma^2$$

$$= \frac{3}{8}\, ma^2\Omega^2\,\underline{j}$$

2 (a) Normal reaction $= mg$ as G not accelerating.

$\therefore$ Couple $= -mga\cos\theta\ \underline{j}$



(b) Gyroscope equation ② $\rightarrow A\dot\Omega_2 + (A\Omega_3 - C\omega_3)\Omega_1 = Q_2 = -mga\cos\theta$

where
$$\begin{cases} \omega_1 = \Omega_1 = -\dot\phi\sin\theta \\ \omega_2 = \Omega_2 = \dot\theta \\ \Omega_3 = \dot\phi\cos\theta \\ \omega_3 = \Omega_3 + \omega \end{cases}$$

No slip at P, so $0 = \underline{v}_P = \underline{v}_G + \underline{\omega} \times a\underline{i}$

But $\underline{v}_G = 0$, $\therefore$ $-\omega_2 a\underline{k} + \omega_3 a\underline{j} = 0$

$\therefore \omega_2 = 0$ and $\omega_3 = 0$

In steady state $\theta =$ constant $\rightarrow \Omega_2 = 0$

So substituting, $A\dot\phi\cos\theta\,.\dot\phi\sin\theta = mga\cos\theta$

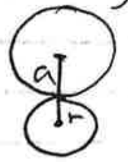
$$\therefore \dot\phi^2 = \frac{mga}{A\sin\theta}$$

where for a disc, $A = \frac{1}{4}ma^2$, $\therefore \dot\phi^2 = \dfrac{4g}{a\sin\theta}$ //

(c) Contact point P rolls round a circle on the table of radius $a\cos\theta$. Viewed in the reference frame in which P stays fixed, this smaller circle rolls against the disc like a gear pair:

Small circle rotates at rate $\dot\phi$, so

Large rotates at rate $-\dfrac{r\dot\phi}{a} = -\dot\phi\cos\theta$



$r = a\cos\theta$

So rotation rate of head, seen in this frame, is $-2\sqrt{\dfrac{g}{a\sin\theta}}\cos\theta$

(i) For θ small, it is natural to view rotation relative to a frame fixed in the table, so rate is $\dot\phi(1-\cos\theta) \simeq \theta^2\sqrt{\dfrac{g}{a\theta}}$

(ii) For θ close to $\pi/2$, more natural to view head relative to vertical axis in frame as in diagram, so rate is $\simeq -2\sqrt{\dfrac{g}{a}}\left(\dfrac{\pi}{2}-\theta\right)$

3 (a) A strap-down system has 3 accelerometers and 3 rate gyros, aligned on orthogonal axes. The whole unit is rigidly fixed in the car/ship/aeroplane etc. The acceleration vector is measured in this vehicle-fixed frame, and before integration it must be "un-rotated" into a universal fixed frame so that the displacement can be formed by cumulative integration along the path travelled.

An accelerometer can be integrated twice to obtain position

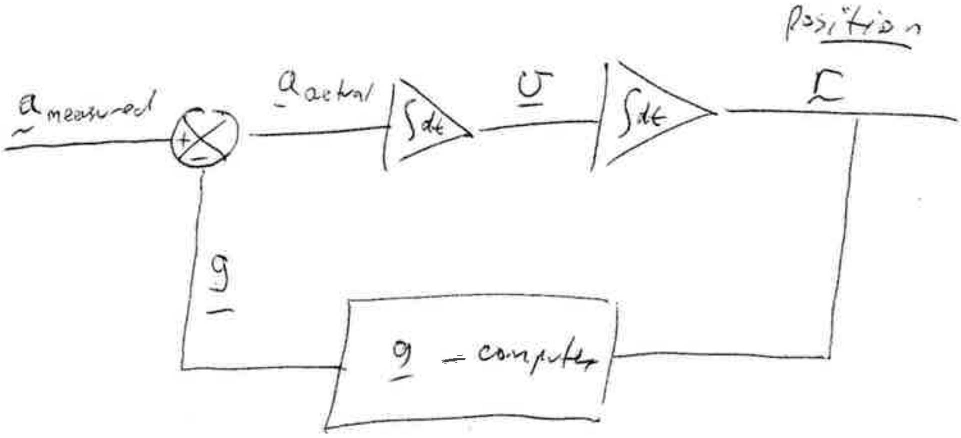
$$a \longrightarrow \int \xrightarrow{\;v\;} \int \longrightarrow x$$

Accelerometers are influenced by gravity $\underset{\sim}{g}$ so three accelerometers alone are insufficient to obtain the displacement of a rigid body (eg ~~[illegible]~~ a satellite). The orientation of the body is measured using three rate gyros, each measuring an angular velocity ω, integrated once to give angle

$$\omega \longrightarrow \int \longrightarrow \theta$$

Euler's angles (or something similar) must be used to resolve finite rotations

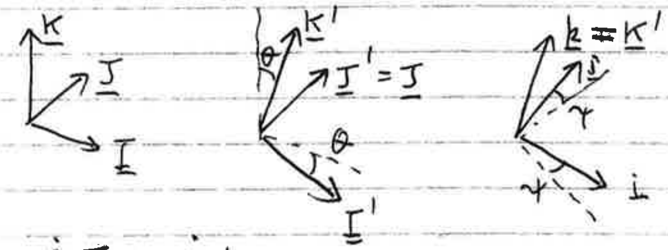
Given position & orientation the local value of $\underline{g}$ can be computed & subtracted from the measured acceleration. The so-called "$\underline{g}$ - computer" is part of a feedback loop as follows



Integration is notoriously sensitive to drift and low-frequency instability so inertial navigation using accelerometers & rate gyros is only useful for short periods of time between recalibration events.

ii) A locally-level platform makes use of the three rate gyros in a feedback system driving torque motors to contrive that the accelerometers are always aligned with $\underline{g}$. This way the computational requirements of the g computer are much reduced ~~and the~~ at the expense of mechanical complexity of the feedback system for the platform

3(b) (i)



$$\underline{w} = \dot{\theta}\,\underline{J} + \dot{\psi}\,\underline{k}$$

But $\underline{J}' = \underline{J} = \underline{j}\cos\psi + \underline{i}\sin\psi$

$$\therefore \underline{w} = \dot{\theta}\sin\psi\,\underline{i} + \dot{\theta}\cos\psi\,\underline{j} + \dot{\psi}\,\underline{k}$$

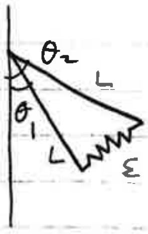
(ii) Now $T = \tfrac{1}{2}A(\dot{\theta}\sin\psi)^2 + \tfrac{1}{2}B(\dot{\theta}\cos\psi)^2 + \tfrac{1}{2}C\dot{\psi}^2$

and there is no potential energy.

So $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{\psi}}\right) - \dfrac{\partial T}{\partial \psi} = 0$

$$\begin{aligned}\therefore\ 0 &= C\ddot{\psi} - A\dot{\theta}^2\sin\psi\cos\psi + B\dot{\theta}^2\cos\psi\sin\psi \\ &= C\ddot{\psi} - (A-B)\dot{\theta}^2\sin\psi\cos\psi \\ &= C\dot{w}_3 - (A-B)\,w_1 w_2 \quad : \text{Euler's equation.}\end{aligned}$$

4(a) By cosine rule, $\varepsilon^2 = L^2 + L^2 - 2L^2\cos(\theta_2-\theta_1)$



So potential energy

$$V = \tfrac{1}{2}k\varepsilon^2 + M_1 g\frac{L}{2}(1-\cos\theta_1) + M_2 g\frac{L}{2}(1-\cos\theta_2)$$

$$= kL^2\left(1-\cos(\theta_2-\theta_1)\right) + \frac{M_1 gL}{2}(1-\cos\theta_1) + \frac{M_2 gL}{2}(1-\cos\theta_2)$$

Kinetic energy $T = \tfrac{1}{2}\left(\tfrac{1}{3}M_1L^2\right)\dot\theta_1^2 + \tfrac{1}{2}\left(\tfrac{1}{3}M_2L^2\right)\dot\theta_2^2$

So Lagrange equations are:

θ_2 : $\tfrac{1}{3}M_2L^2\ddot\theta_2 + kL^2\sin(\theta_2-\theta_1) + M_2 g\frac{L}{2}\sin\theta_2 = 0$

as given

θ_1 : $\tfrac{1}{3}M_1L^2\ddot\theta_1 - kL^2(\sin\theta_2-\theta_1) + M_1 g\frac{L}{2}\sin\theta_1 = 0$

(b) θ_1, θ_2 small so replace $\sin\theta_1$ with θ_1 etc.

Then

$$\underbrace{\begin{bmatrix}\tfrac{1}{3}M_1L^2 & 0\\ 0 & \tfrac{1}{3}M_2L^2\end{bmatrix}}_{\text{Mass matrix } M}\begin{bmatrix}\ddot\theta_1\\ \ddot\theta_2\end{bmatrix} + \underbrace{\begin{bmatrix}\tfrac{1}{2}M_1gL + kL^2 & -kL^2\\ -kL^2 & \tfrac{1}{2}M_2gL + L^2k\end{bmatrix}}_{\text{Stiffness matrix } K}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} = 0$$

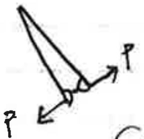
Natural frequencies ω satisfy $\det(K-\omega^2M)=0$

$\therefore \left[M_1\left(-\tfrac{1}{3}L^2\omega^2 + \tfrac{1}{2}gL\right) + kL^2\right]\left[M_2\left(-\tfrac{1}{3}L^2\omega^2 + \tfrac{1}{2}gL\right) + kL^2\right] - (kL^2)^2 = 0$

$\therefore \left(-\tfrac{1}{3}L^2\omega^2 + \tfrac{1}{2}gL\right)\left[M_1M_2\left(-\tfrac{1}{3}L^2\omega^2 + \tfrac{1}{2}gL\right) + kL^2(M_1+M_2)\right] = 0$

So roots are $\omega^2 = \dfrac{3g}{2L}$, $\omega^2 = \dfrac{kL^2(M_1+M_2) + \tfrac{1}{2}gLM_1M_2}{\tfrac{1}{3}M_1M_2L^2}$

(c) Contact gives equal and opposite force $P(t)$ on both rods



For a simple model could be represented as a (stiff) linear "contact spring", or more realistically as a non-linear Hertzian spring.

Generalised forces by virtual work: $\delta W = -P\delta\theta_1 L + P\delta\theta_2 L$

so $Q_{\theta_1} = -PL$, $Q_{\theta_2} = PL$

For linear spring S, $P = -SL(\theta_2-\theta_1)$ when θ_2 goes below θ_1, 0 otherwise

3C5 2012 Answers

1 (a) $\frac{4}{3}ma^2$, $2ma^2$, $2ma^2$

(b) $ma^2\begin{bmatrix} 2/3 & 0 & -3/8 \\ 0 & 1 & 0 \\ -3/8 & 0 & 1 \end{bmatrix}$

(c) $\frac{3}{8}ma^2\Omega^2\mathbf{j}$

2 (a) $-mga\cos\theta\,\mathbf{j}$

(b) $\dot{\phi} = 2\sqrt{\frac{g}{a\sin\theta}}$

(c) (i) $\sqrt{\frac{\theta^3 g}{a}}$ relative to a frame *fixed in the table*

(ii) $-2\sqrt{\frac{g}{a}}\left(\frac{\pi}{2}-\theta\right)$ relative to a frame in which *contact point with table stays fixed*

4 (a) $\frac{1}{3}M_1L^2\ddot{\theta}_1 + \frac{1}{2}M_1gL\sin\theta_1 - kL^2\sin(\theta_2-\theta_1) = 0$

(b) $M = \frac{L^2}{3}\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$, $K = \begin{bmatrix} \frac{1}{2}M_1gL + kL^2 & -kL^2 \\ -kL^2 & \frac{1}{2}M_2gL + kL^2 \end{bmatrix}$

$\omega^2 = \frac{3g}{2L}$, $\frac{kL^2(M_1+M_2) + gLM_1M_2/2}{M_1M_2L^2/3}$