ENGINEERING PART IIA, MODULE 3C6, 2012.

1 (a) Using energy expression from Data Sheet, Rayleigh quotient is

$$\omega_1^2 \simeq \frac{\frac{1}{2} P \int w'^2 dx}{\frac{1}{2} m \int w^2 dx} \quad \text{where } w = \sin\frac{\pi x}{L} \text{ and } m = \pi a^2 \rho$$

$$\therefore \omega_1^2 \simeq \frac{P \int_0^L \left(\frac{\pi}{L}\right)^2 \cos^2 \pi x/L \, dx}{m \int_0^L \sin^2 \pi x/L \, dx} = \frac{P}{m}\left(\frac{\pi}{L}\right)^2 \frac{L/2}{L/2} \qquad \frac{P\pi}{8a^2L^2}$$

$$\therefore \omega_1 \simeq \frac{\pi}{L}\sqrt{\frac{P}{m}} = \frac{\pi}{L}\sqrt{\frac{P}{\pi a^2 \rho}}$$

(b) If stress is $\sigma$, $P = \sigma \pi a^2$

$$\therefore \omega_1 = \frac{\pi}{L}\sqrt{\frac{\sigma \pi a^2}{\rho \pi a^2}} = \frac{\pi}{L}\sqrt{\frac{\sigma}{\rho}} \quad \text{independent of } a$$

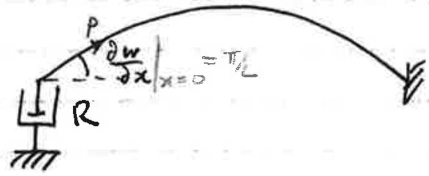
Max $\sigma$ is $\sigma_f$, so max $\omega_1 = \frac{\pi}{L}\sqrt{\frac{\sigma_f}{\rho}}$

or in Hz, max $f_1 = \frac{1}{2L}\sqrt{\frac{\sigma_f}{\rho}}$ independent of a

(c) Force on dashpot

$$f(t) \simeq P \left.\frac{\partial w}{\partial x}\right|_{x=0} = \frac{P\pi}{L}\sin\omega_1 t$$

for $w = \sin\frac{\pi x}{L}\sin\omega_1 t$




Velocity response of dashpot $v = \frac{f}{R}$, so power

dissipated $= vf = \frac{f^2}{R} = \frac{1}{R}\left(\frac{P\pi}{L}\right)^2 \sin^2\omega_1 t$

Total over one period is $D = \frac{1}{R}\left(\frac{P\pi}{L}\right)^2 \frac{1}{2} \times \underbrace{\text{period}}_{= 2\pi/\omega_1}$

$$= \frac{1}{2R}\left(\frac{P\pi}{L}\right)^2 . 2\pi . \frac{L}{\pi}\sqrt{\frac{\pi a^2 \rho}{P}} = \frac{L}{R}\left(\frac{P\pi}{L}\right)^2 \sqrt{\frac{\pi a^2 \rho}{P}}$$

Total P.E. of string from (a), $V = \frac{1}{2} P \left(\frac{\pi}{L}\right)^2 . L/2$

$$\therefore \frac{D}{V} = \frac{\frac{L}{R}\left(\frac{P\pi}{L}\right)^2 \sqrt{\frac{\pi a^2 \rho}{P}}}{\frac{PL}{4}\left(\frac{\pi}{L}\right)^2} = \frac{4}{R}\sqrt{\pi a^2 \rho P}$$

Relevant issues to include in discussion:
For musical purposes, there is a minimum usable tension otherwise the string rattles and buzzes. There is a maximum tension because the string breaks: in practice, tend to operate not too far from the maximum.
There is a minimum length of a string. If there is only one string per note as in a piano, this is to do with sound quality and having enough room for the excitation mechanism. For a guitar, need to have room to finger the string at different lengths to change the note.
The dashpot $R$ represents energy loss into the instrument body, and a fraction of that comes out as radiated sound: to the energy loss ratio is a measure, roughly, of loudness of the sound. So if a given instrument has roughly the same $R$ for each string, loudness scales with the "impedance ratio" $a\sqrt{P\rho}/R$ so heavier/tighter strings are louder.

Piano: separate string for each note. To keep loudness the same over the range, it would be easiest to use identical strings and vary only the length. But you double the length for each octave, and a piano has about 7 octaves. If the minimum length is around 0.1 m, the longest would be over 6 m long, not practical. So have to keep lower strings short, so they have to be heavier. That tends to make them louder. Can't reduce the tension too much, so in practice use 1 string per note low down, then 2 per note in the mid range, then 3 per note higher up.

Guitar: Have to vary the weight and/or tension to achieve the tuning range: but only 2 octaves for open strings of a guitar. The top string is likely to be tuned near the limit for the chosen string material. Lower strings are heavier, but if tension is kept the same they will also be louder, so tend to reduce the tensions a bit to keep the impedance ratios closer.

2(a) Beam equation is $\quad m\dfrac{\partial^2 w}{\partial t^2} + EI\dfrac{\partial^4 w}{\partial x^4} = 0$

For modes, try $w = u(x)\, e^{i\omega t} \rightarrow u'''' - \alpha^4 u = 0$
with $\alpha^4 = \dfrac{m\omega^2}{EI}$

General solution $u = A\cos\alpha x + B\sin\alpha x + C\cosh\alpha x + D\sinh\alpha x$

At $x=0$: $\begin{cases} u=0 \rightarrow A+C=0 \\ u''=0 \rightarrow -\alpha^2 A + \alpha^2 C = 0 \end{cases}$

so $A = C = 0$

At $x=L$: $\begin{cases} u=0 \rightarrow B\sin\alpha L + D\sinh\alpha L = 0 \\ u''=0 \rightarrow -\alpha^2 B\sin\alpha L + \alpha^2 D\sinh\alpha L = 0 \end{cases}$

So $D\sinh\alpha L = 0$ and $B\sin\alpha L = 0$

$\alpha \neq 0$, so $D = 0$ | Don't want $B=0$ or get $u=0$, so must have $\sin\alpha L = 0$

$\therefore \alpha L = n\pi$, $n = 1, 2, 3\ldots$, $\therefore \omega^2 = \dfrac{EI}{m}\left(\dfrac{n\pi}{L}\right)^4$

so natural frequencies $\omega_n = \left(\dfrac{EI}{m}\right)^{1/2} \left(\dfrac{n\pi}{L}\right)^2$

Mode shape $u_n(x) = \sin\alpha x = \sin\dfrac{n\pi x}{L}$

(b) With tension, total restoring shear force on an element of the beam is the sum of the bending term from the beam plus the tension effect as in a string. (Equivalently, potential energy is the sum of bending and tension energies). So equation of motion simply sums the two terms:

$$m\frac{\partial^2 w}{\partial t^2} - P\frac{\partial^2 w}{\partial x^2} + EI\frac{\partial^4 w}{\partial x^4} = 0 \qquad (1)$$

Try $w_n = \sin\dfrac{n\pi x}{L}\, e^{i\omega_n t}$ :

$$(1) \rightarrow \left[-m\omega_n^2 + P\left(\frac{n\pi}{L}\right)^2 + EI\left(\frac{n\pi}{L}\right)^4\right]\sin\frac{n\pi x}{L}\, e^{i\omega_n t} = 0$$

so equation still satisfied provided

$$\omega_n^2 = \frac{P}{m}\left(\frac{n\pi}{L}\right)^2 + \frac{EI}{m}\left(\frac{n\pi}{L}\right)^4$$

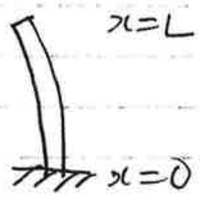
Boundary conditions unchanged from (a), so still bound to work.

(c) If $P < 0$ (compression), $\omega_n$ is reduced by $P$ term. Eventually, $\omega_1^2$ reaches $0$ and then goes negative. Solutions are then $e^{i\omega t}$ with $\omega^2 = -\lambda^2$ say, ie $e^{\pm\lambda t}$. So motion is unstable, growing as $e^{\lambda t}$. This is buckling, and the beam moves in the shape of the corresponding mode, $u_1(x)$


Threshold value of $P$ given by $\omega_1^2 = 0$

$\therefore \quad P_{thr} = -EI\left(\frac{\pi}{L}\right)^2$, $-$ sign for compression.

(d) Same argument applied to column:




Self-weight compressive force $\propto L - x$ so equation is more complicated. But same logic: compressive force reduces each $\omega_n$ (via Rayleigh quotient: potential energy contribution is negative). When first frequency reaches zero, column buckles in shape $u_1(x)$, roughly as sketched.

(e) Columns in a car park are supposed to be identical, so should all have the same lowest natural frequency. If one is carrying extra load its frequency will be lower, approaching zero if close to buckling. So simple NDT procedure is to tap each column with a soft hammer, record the sound, FFT to find the first natural frequency and look for columns that fall below a chosen threshold.

3.(a) Eq's of motion:

$$\left.\begin{aligned}
I_1\ddot{\theta}_1 &= s(\theta_2-\theta_1)\\
I_2\ddot{\theta}_2 &= s(\theta_1-\theta_2)+k(\theta_3-\theta_2)\\
J\ddot{\theta}_3 &= k(\theta_2-\theta_3)+k(\theta_4-\theta_3)\\
&\vdots\\
J\ddot{\theta}_6 &= k(\theta_5-\theta_6)
\end{aligned}\right\} \text{———} (1)$$

In matrix form, $[M]\ddot{\underline{y}} + [K]\underline{y} = \underline{0}$ —— (2)

where $\underline{y} = [\theta_1 \;\; \theta_2 \;\; \cdots \;\; \theta_6]^T$ and

$$[M] = \begin{bmatrix} I_1 & & & & & \\ & I_2 & & & 0 & \\ & & J & & & \\ & & & J & & \\ & 0 & & & J & \\ & & & & & J \end{bmatrix} \qquad [K] = \begin{bmatrix} s & -s & & & & \\ -s & s+k & -k & & 0 & \\ & -k & 2k & -k & & \\ & & -k & 2k & -k & \\ & 0 & & -k & 2k & -k \\ & & & & -k & k \end{bmatrix}$$

(3)

(b) $\underline{y} = \underline{u}e^{i\omega t}$

So (2) becomes $-\omega^2[M]\underline{u}e^{i\omega t} + [K]\underline{u}e^{i\omega t} = \underline{0}$

ie $\left([K]-\omega^2[M]\right)\underline{u} = 0$ —— (4)

Rigid body mode is $\omega = 0$ & $\underline{u}^{(1)} = [1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1]^T$ (5)

Substitute (5) and (3) into (4):

$$[K] \rightarrow \begin{bmatrix} s & -s & & & & \\ -s & s+k & -k & & & \\ & -k & 2k & -k & & \\ & & -k & 2k & -k & \\ & & & -k & 2k & -k \\ & & & & -k & k \end{bmatrix}\begin{Bmatrix} 1\\1\\1\\1\\1\\1 \end{Bmatrix} = \underline{0}$$

As long as all the rows of $[K]$ sum to zero, this is true --- which it is!

QED

3(c) Using measured mode shape:

$\underline{u} = [1 \quad -0.281 \quad -0.284 \quad ----]^T$ from Question

Substitute mode shape into (4) [alternatively use Rayleigh's Quotient]

$$\begin{bmatrix} s-\omega^2 I_1 & -s & & \\ -s & s+k-\omega^2 I_2 & -k & \\ & -k & 2k-\omega^2 J & \\ & & & \ddots \end{bmatrix} \begin{Bmatrix} 1 \\ -0.281 \\ -0.284 \\ -0.286 \\ \vdots \end{Bmatrix} = 0$$

First row: $(s-\omega^2 I_1) - s(-0.281) = 0$

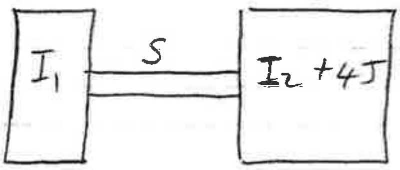
$$\Rightarrow \omega^2 = \frac{s(1+0.281)}{I_1} = \frac{(1.53\times10^6)(1.281)}{2715}$$

$\omega^2 = 722 \ (\text{rad/s})^2$

$\Rightarrow \omega = 26.87$ rad/s $\quad (4.276\ \text{Hz})$

(d) Using approximate mode shape:

The engine $(4J + I_2)$ swings as a rigid unit against the generator $(I_1)$




$$I_1\ddot{\theta}_1 = s(\theta_2 - \theta_1)$$

$$(I_2+4J)\ddot{\theta}_2 = s(\theta_1 - \theta_2)$$

ie $\begin{bmatrix} I_1 & 0 \\ 0 & I_2+4J \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} s & -s \\ -s & s \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \underline{0}$

with the eigenvalue problem $\begin{bmatrix} s-\omega^2 I_1 & -s \\ -s & s-\omega^2(I_2+4J) \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = 0$ (6)

$\Rightarrow (s-\omega^2 I_1)(s-\omega^2(I_2+4J)) - s^2 = \underline{0}$

$\Rightarrow \omega^4 I_1(I_2+4J) - \omega^2(I_1 s + (I_2+4J)s) = 0$

$$\Rightarrow \omega^2\left(\omega^2 - \frac{(I_1+I_2+4J)s}{I_1(I_2+4J)}\right) = 0$$

$\Rightarrow \omega^2 = 0$ (rigid body) and $\omega^2 = \dfrac{(I_1+I_2+4J)s}{I_1(I_2+4J)} = 722.2$

$\Rightarrow \underline{\underline{\omega = 26.87 \text{ rad/s}}} \quad (4.277\ \text{Hz})$

3 (d) cont

This estimated frequency is very similar to the exact value in (c). When calculated in this way, the two eigenvectors are orthogonal ie rigid body mode and $I_1$ vs $I_2+4J$ mode

Check whether these two modes are orthogonal by verifying:

$$\underline{u}^{(1)T}[M]\,\underline{u}^{(2)} = 0 \qquad \text{——— } (7)$$

Find $\underline{u}^{(2)}$ by substituting the e-value into (6)

First row of (6) is:

$$\left[\cancel{S} - \left(\frac{I_1+I_2+4J}{\cancel{I_1}(I_2+4J)}\right)\cancel{I_1}\right]\theta_1 - \cancel{S}\,\theta_2 = 0$$

$$\therefore \frac{\theta_2}{\theta_1} = 1 - \left(\frac{I_1+I_2+4J}{I_2+4J}\right) = \frac{-I_1}{I_2+4J}$$

$$= -0.2816$$

So (7) gives

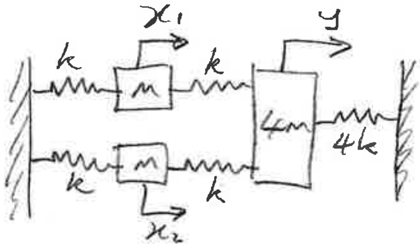
$$[1 \quad 1]\begin{bmatrix} I_1 & 0 \\ 0 & I_2+4J \end{bmatrix}\begin{Bmatrix} 1 \\ -I_1/(I_2+4J) \end{Bmatrix}$$

$$= [1 \quad 1]\begin{Bmatrix} I_1 \\ -\frac{I_1\,\cancel{(I_2+4J)}}{\cancel{I_2+4J}} \end{Bmatrix} = I_1 - I_1 = \underline{\underline{0}}$$

QED

(e) Discussion of damping / friction, idealisation as a disc, gas pressure forces, unbalance, etc.

4. (a) Potential Energy:




$$V = \tfrac{1}{2}4ky^2 + \tfrac{1}{2}kx_1^2 + \tfrac{1}{2}kx_2^2 + \tfrac{1}{2}k(y-x_1)^2 + \tfrac{1}{2}k(y-x_2)^2$$

$$= \tfrac{1}{2}\{6ky^2 + 2kx_1^2 + 2kx_2^2 - 2kyx_1 - 2kyx_2\}$$

$$\therefore = \tfrac{1}{2}[x_1 \; x_2 \; y] \underbrace{\begin{bmatrix} 2k & 0 & -k \\ 0 & 2k & -k \\ -k & -k & 6k \end{bmatrix}}_{[K]} \begin{Bmatrix} x_1 \\ x_2 \\ y \end{Bmatrix} \quad //$$

Kinetic Energy:

$$T = \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}m\dot{x}_2^2 + \tfrac{1}{2}(4m)\dot{y}^2$$

$$= \tfrac{1}{2}[\dot{x}_1 \; \dot{x}_2 \; \dot{y}] \underbrace{\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 4m \end{bmatrix}}_{[M]} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y} \end{Bmatrix}$$

(b) $([K] - \omega^2[M])\underline{u} = 0$

$$\begin{bmatrix} 2k-\omega^2 m & 0 & -k \\ 0 & 2k-\omega^2 m & -k \\ -k & -k & 6k-4\omega^2 m \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ y \end{Bmatrix} = \underline{0} \quad \text{—— (1)}$$

$\Rightarrow$ det $= 0$ : $(2k-\omega^2 m)\left[(2k-\omega^2 m)(6k-4\omega^2 m) - k^2\right] - k[+k(2k-\omega^2 m)]$

$\Rightarrow (2k-\omega^2 m)[12k^2 - 6\omega^2 km - 8\omega^2 mk + 4\omega^4 m^2 - 2k^2] = 0$

$\Rightarrow (2k-\omega^2 m)[4\omega^4 m^2 - 14\omega^2 mk + 10k^2] = 0$

$\Rightarrow \quad \omega^2 = 2k/m \quad \& \quad \omega^2 = \frac{5}{2}k/m, \quad \omega^2 = k/m$

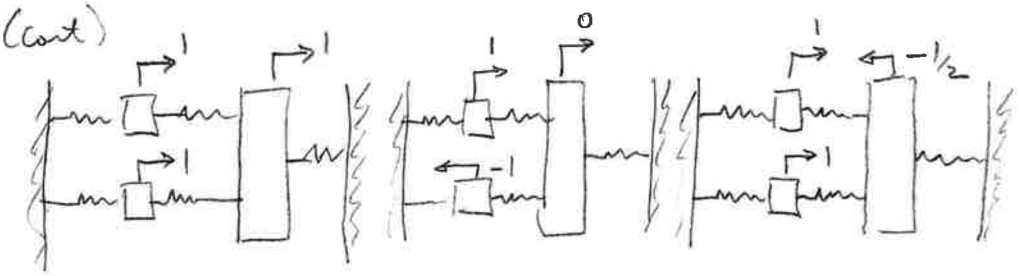
<u>Mode shapes:</u> First row of (1) : $(2k - \omega^2 m)x_1 - ky = 0$

$\omega_1^2 = k/m$: $(2k-k)x_1 - ky = 0 \Rightarrow x_1 = y$ & Row 2: $x_2 = y$
$\Rightarrow [1 \; 1 \; 1]^T$

$\omega_2^2 = 2k/m$: $(0)x_1 - y = 0 \Rightarrow y = 0$ & Row 3: $x_1 = -x_2$
$\Rightarrow [1 \; -1 \; 0]^T$

$\omega_3^2 = \frac{5k}{2m}$ : $(2 - 5/2)x_1 - y = 0 \Rightarrow y = -\frac{x_1}{2}$ & Row 2: $y = -x_2/2$
$\Rightarrow [1 \; 1 \; -\tfrac{1}{2}]^T$

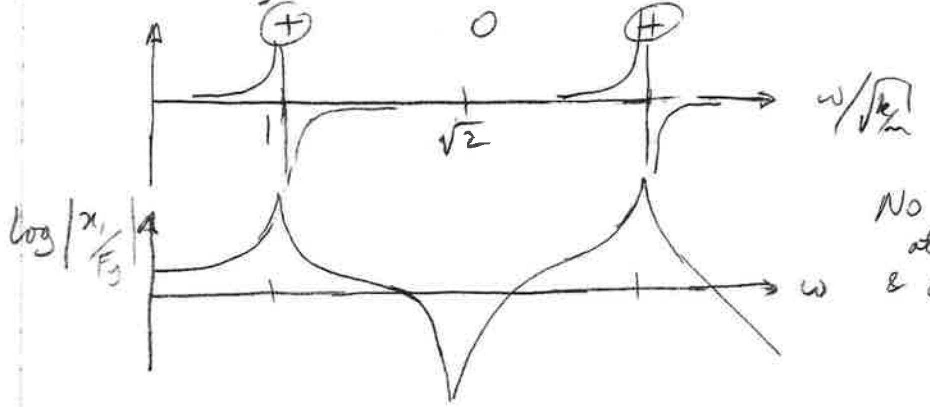
4 (cont)




$\omega_1^2 = k/m$ $\qquad$ $\omega_2^2 = 2k/m$ $\qquad$ $\omega_3^2 = \frac{5}{2}k/m$

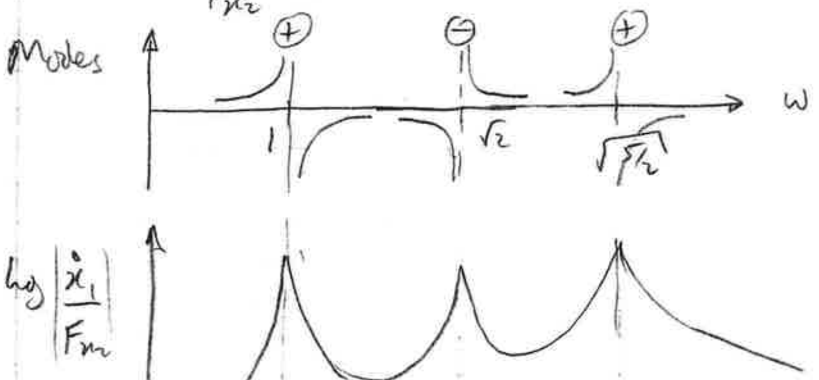
4(c) (i) $\dfrac{x_1}{F_y}$ : $\qquad H(\omega) = \sum \dfrac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 - \omega^2}$

$u_j \, u_k$




No motion at $\omega_2$ & anti-resonance

4c (ii) $\dfrac{\dot{x}_1}{F_{x_2}}$ :




No anti-resonances