2011/12 3C7 20/12/11

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Q1

$$\frac{\partial \phi}{\partial r} = \left(-\frac{a^2 r^{-1}}{2} + \frac{r}{2}\right)\sigma_0 + \left(-\frac{r}{2} + \frac{a^4 r^{-3}}{2}\right)\sigma_0 \cos\theta$$

$$\frac{\partial^2 \phi}{\partial r^2} = \left(\frac{a^2 r^{-2}}{2} + \frac{1}{2}\right)\sigma_0 + \left(-\frac{1}{2} - \frac{3a^4 r^{-4}}{2}\right)\sigma_0 \cos\theta$$

$$\frac{1}{r}\frac{\partial \phi}{\partial \theta} = -2\sigma_0 \sin 2\theta \left(\frac{r}{4} - \frac{a^4 r^{-3}}{4} + \frac{a^2 r^{-1}}{2}\right)$$

$$\frac{\partial^2 \phi}{\partial \theta^2} = +4\sigma_0 \cos 4\theta \left(\frac{r^2}{4} + \frac{a^4 r^{-2}}{4} - \frac{a^2}{2}\right)$$

$$\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right\} = -2\sigma_0 \sin 2\theta \left(\frac{1}{4} + \frac{3a^4 r^{-4}}{4} - \frac{a^2 r^{-2}}{2}\right)$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$= \frac{\sigma_0}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{\sigma_0 \cos 2\theta}{2}\left(1 - \frac{4a^2}{r^2} + \frac{3a^4}{r^4}\right) \checkmark$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} = \frac{\sigma_0}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma_0}{2}\cos 2\theta \left(1 + \frac{3a^4}{r^4}\right) \checkmark$$

$$\sigma_{r\theta} = -\frac{\sigma_0 \sin 2\theta}{2}\left(1 + \frac{2a^2}{r^2} - \frac{3a^4}{r^4}\right) \checkmark$$

(b) Elastic soln: must satisfy compatibility $\Rightarrow \nabla^4\phi = 0$
(given) ✓

must satisfy 'internal' equil
(satisfied by Airy) ✓

must satisfy B.C.'s

B.C.

At $r = a$, $\sigma_{rr} = 0$, $\sigma_{r\theta} = 0$

$\sigma_{rr} = 0$ ✓
$\sigma_{r\theta} = 0$ ✓

At $r \to \infty$

$\sigma_{rr} = \frac{\sigma_0}{2}(1 + \sigma_0 \cos\theta)$

$\sigma_{\theta\theta} = \frac{\sigma_0}{2}(1 - \sigma_0 \cos\theta)$

$\sigma_{r\theta} = -\sigma_0 \frac{\sin 2\theta}{2}$

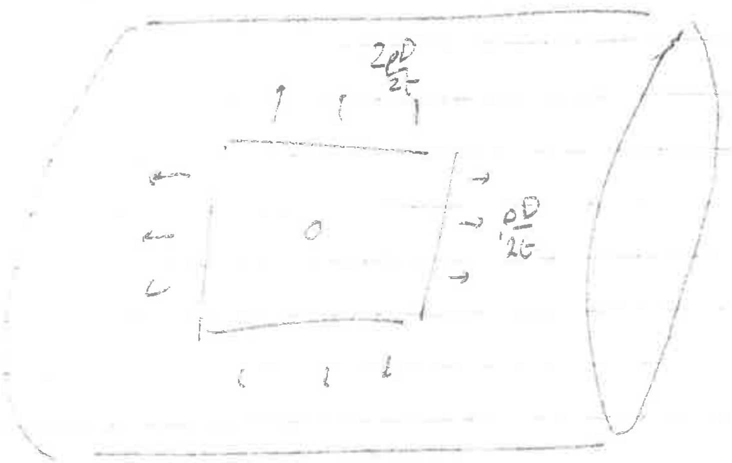
i.e., $\theta = 0$

$\sigma_{rr} = \sigma_0$
$\sigma_{\theta\theta} = 0$
$\tau_{r\theta} = 0$

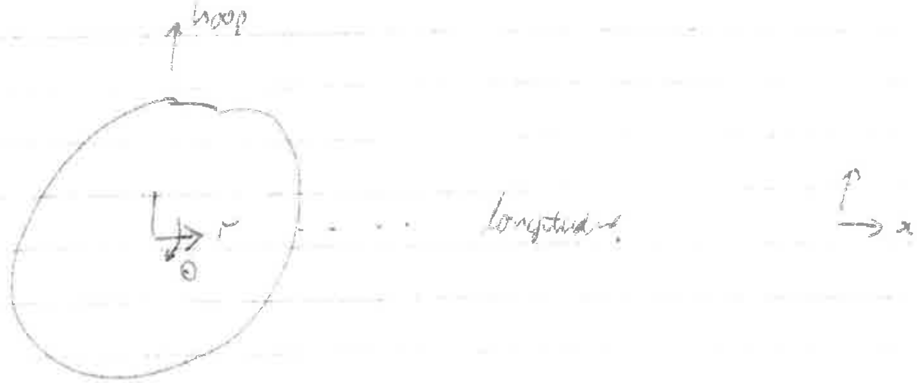
$\theta = \pi/2$, $\sigma_{rr} = 0$
$\sigma_{\theta\theta} = \sigma_0$
$\sigma_{r\theta} = 0$

i.e. uniaxial stress ✓

(c)



Superpose two uniaxial solutions



Consider longitudinal direction, $\theta = 0$

Due to longitudinal stress

$$\sigma_{yy} = \sigma_{\theta\theta} = \frac{\sigma_0}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma_0}{2}\left(1 + \frac{3a^4}{r^4}\right)$$

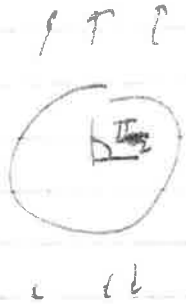
$$= \frac{\sigma_0}{2}\left(\frac{a^2}{r^2} - \frac{3a^4}{r^4}\right)$$

$$= \cancel{\frac{2pD}{}}\ \cancel{\frac{pD}{2}}$$

$$\sigma_0 = \frac{pD}{2t}$$

$$\sigma_{yy} = \frac{pD}{4t}\left(\frac{a^2}{r^2} - \frac{3a^4}{r^4}\right)$$

Due to hoop stress



$$\sigma_{yy} = \sigma_{\theta\theta} = \frac{\sigma_0}{2}\left(1 + \frac{a^2}{r^2}\right) + \frac{\sigma_0}{2}\left(1 + \frac{3a^4}{r^4}\right)$$

$$= \frac{\sigma_0}{2}\left(2 + \frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right)$$

$$\sigma_0 = \frac{2pD}{2t}$$

$$\sigma_{yy} = \frac{2pD}{4t}\left(2 + \frac{a^2}{r^2} + \frac{3a^4}{r^4}\right)$$

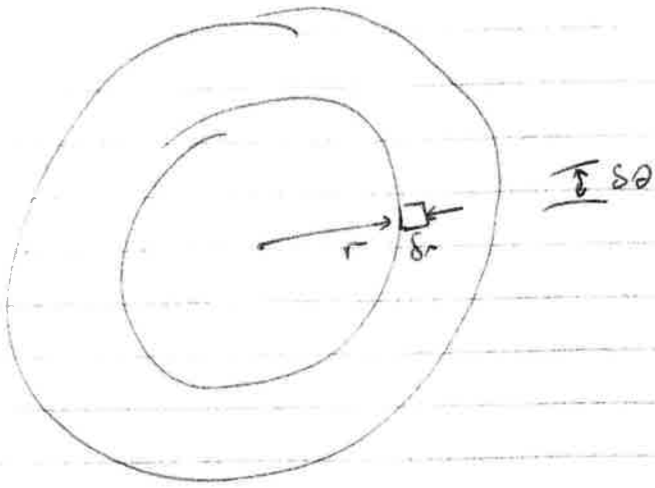
Total

$$\sigma_{yy} = \frac{pD}{4t}\left(\cancel{2}\,4 + 3\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right)$$

~~If σ_{yy}~~ Check Check, $r \to \infty$, $\sigma_{yy} \to \frac{pD}{t}$ ✓

$r = a$ $\sigma_{yy} = \frac{10pD}{4t} = 3 \times \frac{2pD}{2t} - \frac{pD}{2t}$ ✓

Q 2.



Torque on element

$$\delta T = \underbrace{G\beta r \,.\, \delta r\, r\delta\theta}_{\text{force}} \,.\, \underbrace{r}_{\text{lever arm.}}$$

Torque on complete circumference

$$\delta T = 2\pi.\, G\beta r^3 .\, \delta r$$

Integrate from 0 to $\frac{D}{2}$

$$T = 2\pi G\beta \int_0^{D/2} r^3\, dr$$

$$= 2\pi G\beta \left[\frac{r^4}{4}\right]_0^{D/2} = \frac{\pi G\beta}{2}\left[r^4\right]_0^{D/2}$$

$$= \cancel{\frac{2\pi G D^3}{24}} = \cancel{\frac{\pi G D^3}{12}} = \frac{\pi G\beta D^4}{32}$$

[check, $J = \frac{\pi}{2}\left(\frac{D}{2}\right)^4 = \frac{\pi D^4}{32}$ $\quad T = GJ\theta' = \frac{\pi G D^4}{32}$ ✓] OK

(b) Prandtl stress function has ~~a~~ magnitude of slope equal to torque

(axisymmetric ⟹ only a function of r)

$$\cancel{\frac{\partial \psi}{\partial r}} \quad \frac{d\psi}{dr} = -G\beta r$$

$$\therefore \psi = -\frac{G\beta r^2}{2} + C$$

choose $\psi = 0$ at boundary

$$\Rightarrow 0 = -\frac{G\beta D^2}{8} + C$$

$$\boxed{\begin{aligned} \therefore \psi &= \frac{G\beta D^2}{8} - \frac{G\beta r^2}{2} \\ &= G\beta\left(\left(\frac{D}{2}\right)^2 - r^2\right) \end{aligned}}$$

$$\nabla^2\psi = \left[\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta^2}\right]\cdot\psi$$

but $\frac{\partial^2\psi}{\partial\theta^2} = 0$

$$\frac{\partial\psi}{\partial r} = -G\beta r$$

$$\frac{\partial^2\psi}{\partial r^2} = -G\beta$$

$$\nabla^2\psi = -G\beta - \cancel{2}\,G\beta = -2G\beta \quad \checkmark$$

$$\int \psi \, dA = \int_0^{D/2} \psi \cdot 2\pi r \, dr$$

$$= \cancel{G\beta D^4}$$

$$= 2\int_0^{D/2} \left[\frac{\pi G\beta}{4} D^2 . r - \pi G\beta \, r^3 \right] dr$$

$$= \left[\frac{\pi G\beta D^2}{8} . r^2 - \frac{\pi G\beta}{4} r^4 \right]_0^{D/2}$$

$$= \pi G\beta D^4 \left(\frac{1}{8} \times \frac{1}{4} - \frac{1}{4} \times \frac{1}{16} \right)$$

$$= \frac{\pi G\beta D^4}{64}$$

$$T = 2\int \psi \, dA$$

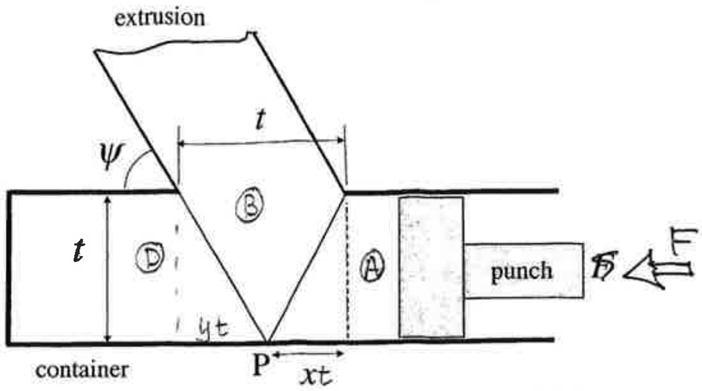
$$= \frac{\pi G\beta D^4}{32}$$ ✓

(c) The Prandtl stress function will not be greatly affected remotely from the slit, and hence the stiffness will be little changed.

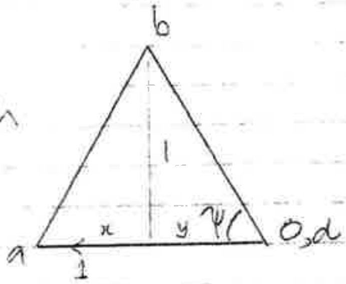
However, at the reentrant corner, the slope of ψ must $\rightarrow \infty$, implying a local stress concentration, and plasticity will occur as soon as torque is applied.

4/3'

Q3



hodograph



Upper Bound $F\,1 = k\,(l_{ab}.v_{ab} + l_{bd}.v_{bd} + f\,x t.1)$

But $l_{ab} = t\sqrt{1+x^2}$ $\quad l_{bd} = t\sqrt{1+y^2}$

$v_{ab} = ab = \sqrt{1+x^2}$ $\quad v_{bd} = \sqrt{1+y^2}$

$\therefore \dfrac{F}{kt} = (1+x^2) + (1+y^2) + fx$

But $y = 1-x$ $\quad 1+y^2 = 2-2x+x^2$

$\therefore F/kt = 1+x^2+2-2x+x^2+fx$

$F/kt = 2x^2 - (2-f)x + 3$

$\dfrac{d(F/kt)}{dx} = 4x-(2-f)$ $\quad \therefore \dfrac{d(F/kt)}{dx} = 0$ when $x = \dfrac{2-f}{4}$

(i) $\therefore$ when $f=0$, $x = 1/2$ as by symmetry, but. $\psi = \tan^{-1} 1/y = \tan^{-1} 2$

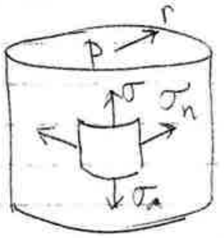
(ii) $f = 0.2$ $\quad x = \dfrac{2-.2}{4} = 0.45$

(iii) $\dfrac{F}{kt} = \dfrac{2(2-f)^2}{16} - \dfrac{(2-f)^2}{4} + 3 = 3 - (2-f)^2/8$

$\therefore$ when $f = 0.4$ $\quad F/kt = 3 - \dfrac{1.6^2}{8} = 2.68$

Q4 Test 1

(a)



$\sigma_h = \frac{pr}{t}$; $\varepsilon_a = 0$; $\sigma_r \Rightarrow 0$

$(p\,2r = \sigma 2t)$

But $E\varepsilon_a = \sigma_a - \nu\sigma_h - \nu\sigma_r$

$\therefore \quad \sigma_a = \nu\sigma_h$

Von Mises $(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 2Y^2$

$\sigma_1, \sigma_2, \sigma_3$ are principal stresses

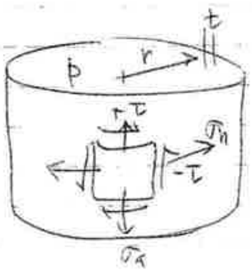
$\sigma_1 = \sigma_h$; $\sigma_2 = \sigma_a$; $\sigma_3 = \sigma_r$

$\sigma_h^2(1-\nu)^2 + \nu^2\sigma_h^2 + \sigma_h^2 = 2Y^2$

$\therefore \quad \sigma_h = \frac{Y}{\sqrt{1-\nu+\nu^2}} = \frac{Y}{\sqrt{1-.3+.09}} = 1.125Y$

$\therefore \quad p = 1.125\,\frac{Yt}{r}$

(b) Test 2



as before $\varepsilon_a = 0 \therefore \sigma_a = \nu\sigma_h$

But now shear stresses as shown

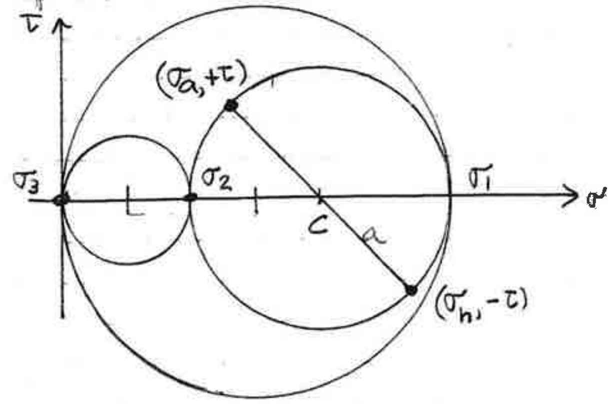
such that $\tau = \frac{T}{2\pi r^2 t}$

Consider Mohr's circle for state of stress shown.

centre C is at $\frac{\sigma_a+\sigma_h}{2} = \frac{1+\nu}{2}\sigma_h$

radius a

$= \left\{\left(1-\frac{1+\nu}{2}\right)^2\sigma_h^2 + \tau^2\right\}^{1/2}$



hence $a^2 = \left(\frac{1-\nu}{2}\right)^2 \sigma_h^2 + \tau^2$ — ①

So that $\sigma_1 = \frac{1+\nu}{2}\sigma_h + a$; $\sigma_2 = \frac{1+\nu}{2}\sigma_h - a$; $\sigma_3 = 0$

So von Mises becomes

$$(2a)^2 + \left[\frac{1+\nu}{2}\sigma_h - a\right]^2 + \left[\frac{1+\nu}{2}\sigma_h + a\right]^2 = 2Y^2$$

i.e. $4a^2 + 2a^2 + 2\left(\frac{1+\nu}{2}\right)^2\sigma_h^2 = 2Y^2$

or $3a^2 + \left(\frac{1+\nu}{2}\right)^2 \sigma_h^2 = Y^2$

now substitute for a from ①

$$3\left(\frac{1-\nu}{2}\right)^2\sigma_h^2 + 3\tau^2 + \left(\frac{1+\nu}{2}\right)^2\sigma_h^2 = Y^2$$

$$3(1-\nu)^2\sigma_h^2 + 12\tau^2 + (1+\nu)^2\sigma_h^2 = 4Y^2$$

$$(4 - 4\nu + 4\nu^2)\sigma_h^2 + 12\tau^2 = 4Y^2$$

$$3\tau^2 = Y^2 - (1-\nu+\nu^2)\sigma_h^2$$

But if $\sigma_h = \frac{1.125}{2}\frac{Yt}{r}\cdot\frac{r}{t} = 0.563Y \qquad \nu = 0.3$

then $\frac{3\tau^2}{Y^2} = 1 - (1 - .3 + .09).563^2$

$\therefore \tau = 0.5Y$

$\therefore T = 2\pi r^2 t\,\tau = 3.14 r^2 t Y$

(c)

$$\sigma_n = 1.125Y$$

$$\sigma_a = 0.3375Y \qquad \Rightarrow \sigma_m = 0.4875Y$$

$$\sigma_t = 0$$

$$\sigma_n' = 0.6375, \quad \sigma_a' = -0.15Y, \quad \sigma_t' = -0.4875Y$$

$$\frac{d\varepsilon_n^p}{\sigma_n'} = \frac{d\varepsilon_a^p}{\sigma_a'} = \frac{d\varepsilon_t^p}{\sigma_t'}$$

$$\Rightarrow \frac{d\varepsilon_n^p}{d\varepsilon_a^p} = -4.25, \qquad \frac{d\varepsilon_a^p}{d\varepsilon_t^p} = 0.3077$$

$$\Rightarrow d\varepsilon_n^p : d\varepsilon_a^p : d\varepsilon_t^p = -4.25 : 1 : 3.25$$

Answers to 3C7: Mechanics of Solids (2011-2012)

1. (c) $\frac{pD}{8t}\left[4+\frac{3a^2}{r^2}+\frac{5a^4}{r^4}\right]$

2. (a) $T=\frac{\pi G\beta D^4}{32}$

   (b) $\psi = G\beta\left[\left(\frac{D}{2}\right)^2 - r^2\right]$

3. (b)(ii) $x = 0.45$

   (b)(iii) $F = 2.68kt$

4. (a) $p = 1.125Yt/r$

   (b) $T = 3.14r^2tY$

   (c) $d\varepsilon_h^p : d\varepsilon_a^p : d\varepsilon_t^p = -4.25 : 1 : 3.25$