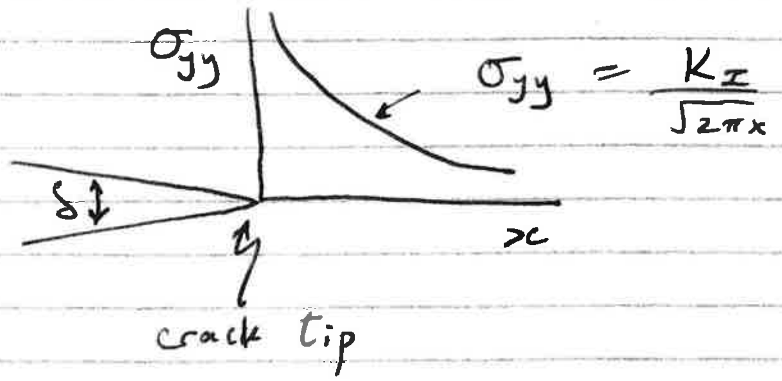
CRIB for 3C9 : Engineering Fracture Mechanics of Materials and Structures

2011/12

1. (a) $G = -\dfrac{\partial \Psi}{\partial A}$ ← potential energy; ∂A ← crack area.

↑ Energy release rate

K characterises the level of stress singularity at a crack tip in a linear elastic solid. Consider mode I loading:



The crack opening $\delta(x)$ also scales with K:

$$\delta \propto \frac{K}{E}\sqrt{|x|}$$

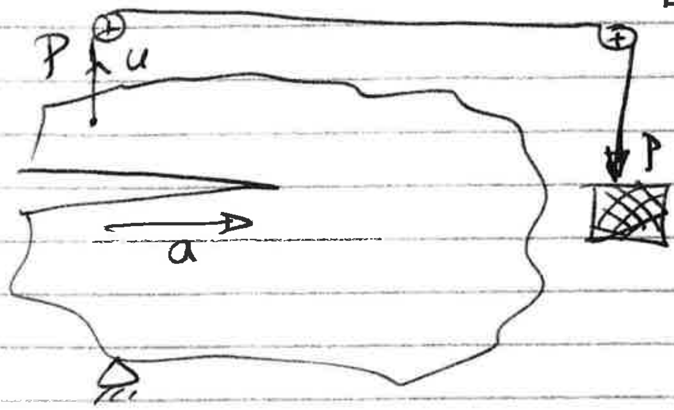
. Similarly for modes II & III.

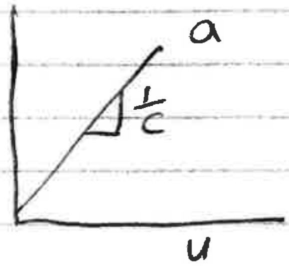
Fracture occurs when $K \to K_{IC}$ or equivalently $G \to G_{IC}$.

$$EG = K^2$$

– the Irwin relation.

1. (b) Prove that $G = \frac{1}{2} \frac{P^2}{B} \frac{\partial C}{\partial a}$





Compliance C is given by

$$C = u/P \qquad P = \frac{u}{C}$$

Ψ = potential energy of the body and loading system.

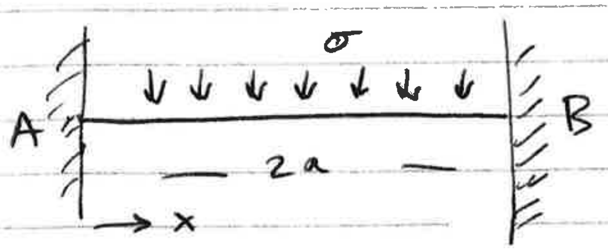
$$\Psi = W - P.u \qquad W = \frac{1}{2} P u = \frac{1}{2} C P^2$$

$$G = -\frac{\partial \Psi}{B \partial a}$$

$$\Rightarrow \quad \Psi = -\frac{1}{2} C P^2 \quad \Rightarrow \quad \frac{\partial \Psi}{\partial a} = -\frac{1}{2} P^2 \frac{\partial C}{\partial a}$$

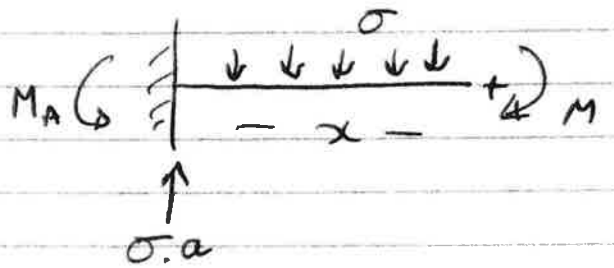
$$\Rightarrow \quad G = \frac{1}{2} \frac{P^2}{B} \frac{\partial C}{\partial a}$$

1. (c)



From Data Book, '$M_A = \frac{wl^2}{12}$' where $l = 2a$
$w = \sigma$

$$\Rightarrow \quad M_A = \frac{\sigma a^2}{3}$$



$$M - \frac{1}{2}\sigma x^2 + \sigma a x - \frac{\sigma a^2}{3} = 0$$

$$\Rightarrow \quad M(x) = \frac{\sigma}{6}\left(3x^2 + 2a^2 - 6ax\right)$$

$$0 \leq x \leq a$$

$$\frac{U}{4} = \int_0^a \frac{1}{2}\frac{M^2}{EI}\,dx$$

$$\Rightarrow \quad U = \frac{2}{EI}\int_0^a \frac{\sigma^2}{36}\left(3x^2 - 6ax + 2a^2\right)^2 dx$$

$$\Rightarrow \quad U = \frac{\sigma^2}{18EI}\int_0^a [9x^4 + 36a^2x^2 + 4a^4 - 36ax^3 - 24a^3x + 12a^2x^2 \; dx$$

$$\Rightarrow \quad U = \frac{\sigma^2}{18EI}\Big[\frac{9}{5}x^5 + 12a^2x^3 + 4a^4x - 9ax^4 - 12a^3x^2 + 4a^2x^3$$

1. (c) (contd.)

$$\Rightarrow \quad \bar{U} = \frac{\sigma^2 a^5}{18 E I}\left(\frac{9}{5} + 16 - 21 + 4\right)$$

$$\Rightarrow \quad \bar{U} = \frac{4}{5}\,\frac{\sigma^2 a^5\, 12}{18 E h^3} = \frac{8}{15}\,\frac{\sigma^2 a^5}{E h^3}$$

$$\left(I = \tfrac{1}{12} h^3\right)$$

Potential energy $\Pi = -\bar{U}$ since the problem is linear.

$$2G = -\frac{\partial \Pi}{\partial a} = +\frac{\partial}{\partial a}\left(\frac{8}{15}\,\frac{\sigma^2 a^5}{E h^3}\right)$$

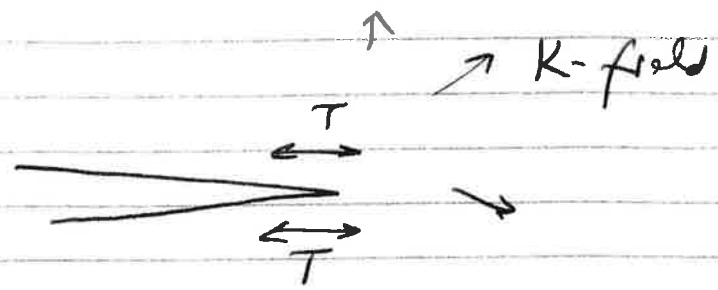
$$= \frac{8}{3}\,\frac{\sigma^2 a^4}{E h^3}$$

$$\Rightarrow \quad G = \frac{4}{3}\,\frac{\sigma^2 a^4}{E h^3}$$

$K_I = \sqrt{EG}$ for plane stress

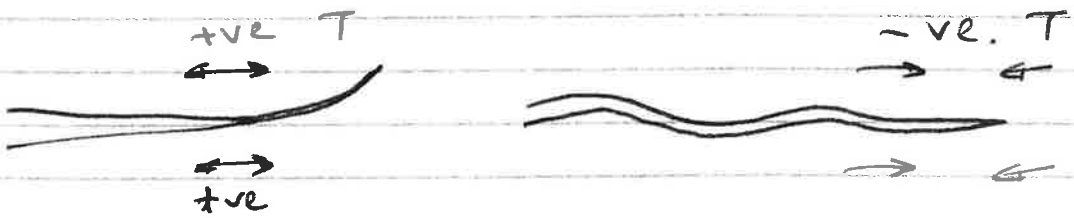
$$\Rightarrow \quad K_I = \frac{2}{\sqrt{3}}\,\frac{\sigma a^2}{h\sqrt{h}}$$

Q2. (a)

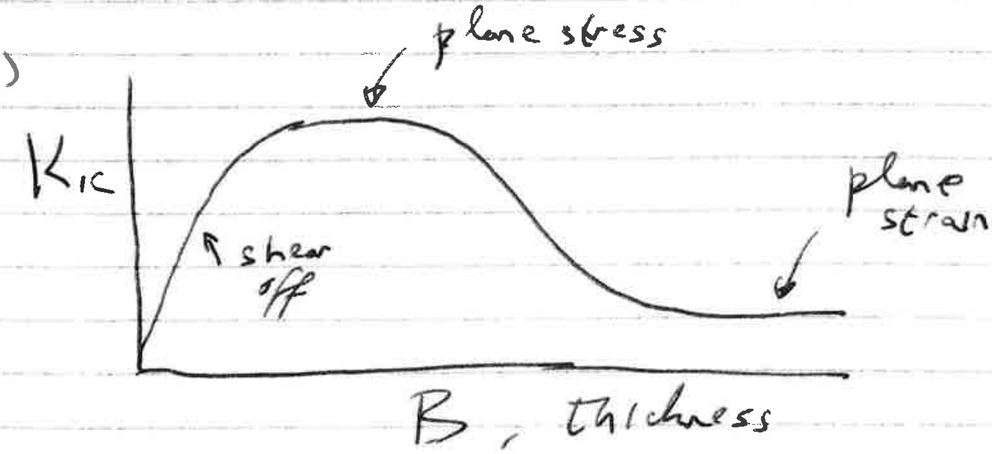


The T-stress is the next highest term for the state of stress near a crack tip. It is independent of the applied K, and has no effect upon the fracture toughness of a ceramic.

A +ve. T-stress destabilises the crack path so that the crack veers away from a straight path. A -ve. T stress stabilises the crack path.



(b)



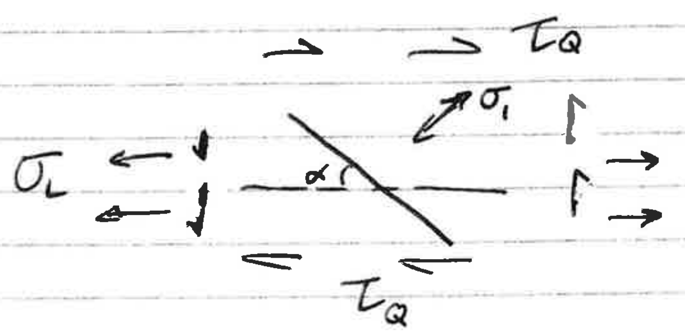
Small sheet thickness B ⇒ plane stress state at the crack tip and limited void growth ⇒ high toughness

Q2 (b) contd.

Large sheet thickness $\left(B > 2.5 \frac{K_{Ic}^2}{\sigma_Y^2} \right)$ will lead to a plane strain state at the crack tip with high stress triaxiality. Hence, the rapid growth of voids and a reduced toughness.

Note, for very thin sheet, shear-off occurs and the toughness G_c scales with the sheet thickness B according to $G_c \sim \sigma_Y B$.

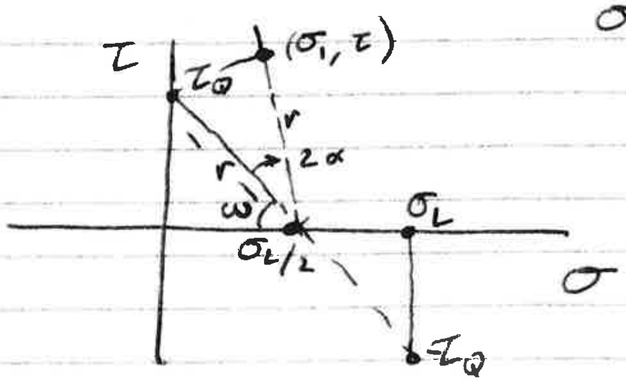
(c)



$$\sigma_L = \frac{P}{2\pi R t}$$

$$\tau_Q = \frac{Q}{2\pi R^2 t}$$

$\sigma_1 = ?$



$$r = \sqrt{\tau_Q^2 + \frac{\sigma_L^2}{4}} \qquad \tan\omega = \frac{2\tau_Q}{\sigma_L}$$

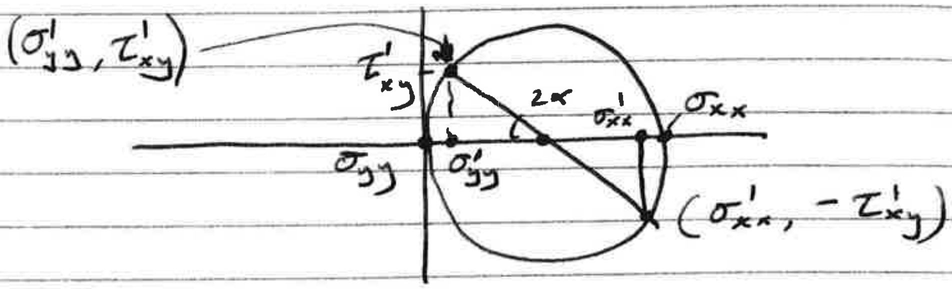
$$\sigma_1 = \frac{\sigma_L}{2} - r\cos(\omega + 2\alpha)$$

$$K_I = \sigma_1 \sqrt{\pi a}$$

Q2 (c)(ii) For case of axial load P only

we have $\sigma_{xx} = \frac{P}{2\pi R t}$ $\quad \sigma_{yy} = 0$

$\tau_{xy} = 0$



By Mohr's ⊙, $\sigma'_{yy} = \frac{\sigma_{xx}}{2}(1 - \cos 2\alpha)$

check : $\alpha = 0 \Rightarrow \sigma'_{yy} = 0$ ✓

$\alpha = \frac{\pi}{2} \Rightarrow \sigma'_{yy} = \sigma_{xx}$ ✓

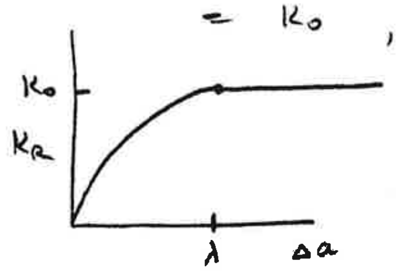
So $K_I = \sigma'_{yy}\sqrt{\pi a}$

$$= \frac{P}{4\pi R t}\sqrt{\pi a}\,(1 - \cos 2\alpha)$$

3. (a)

$$K_R = K_0 \sin \frac{\pi \Delta a}{2 \lambda} \quad , \quad 0 \le \Delta a \le \lambda$$

$$= K_0 \quad , \quad \Delta a > \lambda$$



$$K = 1.12\, \sigma^{\infty} \sqrt{\pi(a_0 + \Delta a)}$$

Equate : $$K = 1.12\sigma^{\infty}\sqrt{\pi\lambda}\left(\frac{a_0}{\lambda} + \frac{\Delta a}{\lambda}\right)^{\frac{1}{2}}$$

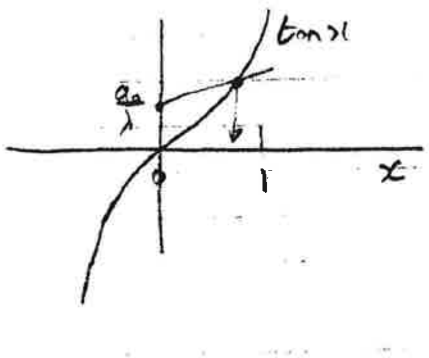
$$= K_0 \sin\left(\frac{\pi}{2}\frac{\Delta a}{\lambda}\right) \qquad -(1)$$

At maximum load, $\frac{\partial K}{\partial a} = \frac{\partial K_R}{\partial a}$

$$\Rightarrow \quad \frac{1.12\,\sigma^{\infty}\sqrt{\pi\lambda}}{2\left(\frac{a_0}{\lambda} + \frac{\Delta a}{\lambda}\right)^{\frac{1}{2}}}\frac{1}{\lambda} = K_0 \frac{\pi}{2\lambda}\cos\left(\frac{\pi}{2}\frac{\Delta a}{\lambda}\right) \qquad -(2)$$

Now divide (1) by (2) $\Rightarrow$

$$\cancel{2\lambda}\left(\frac{a_0}{\lambda} + \frac{\Delta a}{\lambda}\right) = \frac{\cancel{2\lambda}}{\pi}\tan\left(\frac{\pi}{2}\frac{\Delta a}{\lambda}\right)$$



i.e. $\frac{a_0}{\lambda} + x = \frac{1}{\pi}\tan\left(\frac{\pi}{2}x\right)$

where $x \equiv \frac{\Delta a}{\lambda}$

3(a) contd. Solve by iteration :

$$\left(\frac{a_0}{\lambda} + \frac{\Delta a}{\lambda}\right) = \frac{1}{\pi} \tan\left(\frac{\pi}{2} \frac{\Delta a}{\lambda}\right)$$

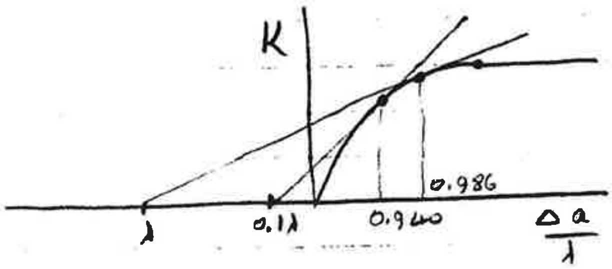
$a_0/\lambda = 1$ $\Rightarrow$ Guess $\frac{\Delta a}{\lambda} = 0.7 \rightarrow 0.882, 0.894$

Converged solution : $\frac{\Delta a}{\lambda} = 0.894$.

$$\Rightarrow \quad K_R = 0.986 \; K_0$$

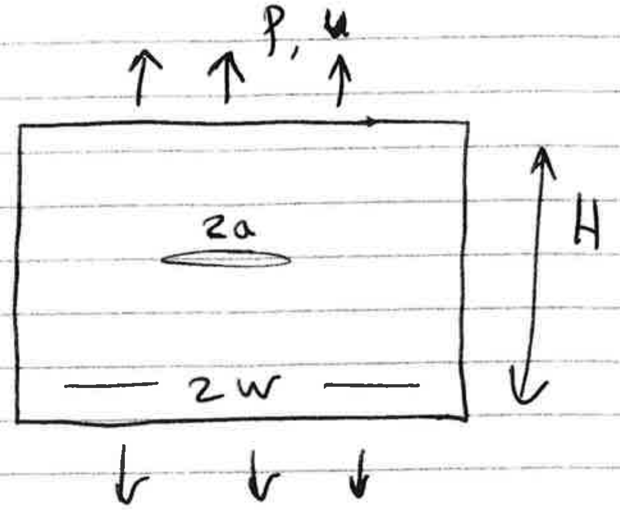
$a_0/\lambda = 0.1$ Guess $\frac{\Delta a}{\lambda} = 0.2 \rightarrow 0.481, 0.681$
$0.754, 0.777$

Converged solution : $\Delta a/\lambda = 0.779$
$K_R = 0.940 \; K_0$.



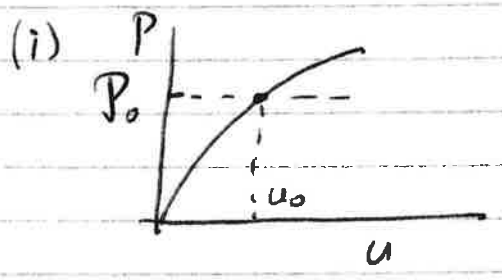
So the sheet fails at a lower K value for the case $a_0 = 0.1\lambda$, and consequently appears to be more brittle.

3(b)



$$P = \sigma_Y B (W-a)\left(\frac{u}{H}\right)^{1/2}$$

(i)



Internal energy $\quad W = \int_0^{u_0} P\, du$

$$\Rightarrow \quad W = \int_0^{u_0} \sigma_Y B (W-a) \frac{u^{1/2}}{H^{1/2}}\, du$$

$$= \frac{2}{3} \frac{u_0^{3/2}}{H^{1/2}} \sigma_Y B (W-a)$$

Potential Energy $\quad \Psi = W - P_0 u_0$

$$\Rightarrow \quad \Psi = -\frac{1}{3} \frac{u_0^{3/2}}{H^{1/2}} \sigma_Y B (W-a) \quad \left(\equiv -\tfrac{1}{2} W\right)$$

$$= -\frac{1}{3}\left[\frac{P_0}{\sigma_Y B (W-a)}\right]^3 H \sigma_Y B (W-a)$$

3. (b) contd.

$$\Rightarrow \text{(i)} \quad \Psi = -\frac{1}{3} \frac{P_o^3 \, H}{\sigma_Y^2 \, B^2 \, (w-a)^2}$$

$$2 J = -\frac{\partial \Psi}{\partial a} = \frac{2}{3} \frac{P_o^3 \, H}{\sigma_Y^2 \, B^2 (w-a)^3}$$

$$\Rightarrow \quad J = \frac{1}{3} \frac{P_o^3 \, H}{\sigma_Y^2 \, B^2 (w-a)^3}$$

4. (a) Initial flaw size is $a_o = 0.1t$

Suppose $\Delta K > \Delta K_{TH}$ at $a = a_o$

i.e. $\Delta\sigma\sqrt{\pi a_o} > \Delta K_{TH}$

$$\Rightarrow \quad \Delta\sigma > \frac{\Delta K_{TH}}{\sqrt{\pi a_o}} \equiv \Delta\sigma_{TH}$$

(b) $$\frac{da}{dN} = C\,\Delta K^n = C\,\Delta\sigma^n (\pi a)^{n/2}$$

$$\Rightarrow \int_{a_o}^{a_f} \frac{da}{a^{n/2}} = C\,\Delta\sigma^n \pi^{n/2} N_f$$

$$\frac{2}{2-n}\left(a_f^{\frac{2-n}{2}} - a_o^{\frac{2-n}{n}}\right) = C\,\Delta\sigma^n \pi^{n/2} N_f$$

$a_f = ?$ If $K_{max} = \sigma_{max}\sqrt{\pi a_f} = K_{IC}$

then $a_f = \frac{1}{\pi}\left(\frac{K_{IC}}{\sigma_{max}}\right)^2$

$$\Delta\sigma = (1-R)\,\sigma_{max}$$

$$\Rightarrow \quad a_f = \frac{1}{\pi}\left[\frac{K_{IC}(1-R)}{\Delta\sigma}\right]^2$$

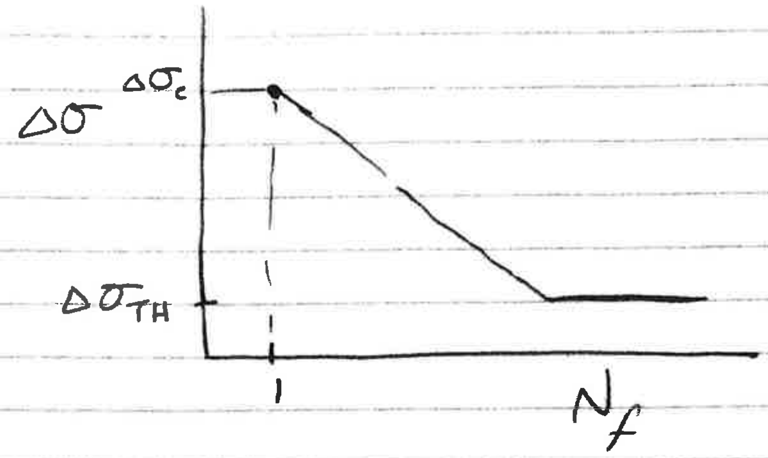
So, ~~with~~ an increase in $\Delta\sigma$ or R
$\Rightarrow$ a reduced a_f $\Rightarrow$ shorter life N_f

If $a_f \gg a_o$ we have

$$\frac{-2}{2-n}\, a_o^{\frac{2-n}{n}} \approx C\,\Delta\sigma^n \pi^{n/2} N_f$$

$$\Rightarrow \quad \Delta\sigma^n N_f = \text{constant}.$$

4 (b) contd.



Specimen fails on 1st. loading cycle when $\sigma_{max}\sqrt{\pi a_0} = K_{IC}$

i.e. $\Delta\sigma = K_{IC}(1-R)\sqrt{\pi a_0} \equiv \Delta\sigma_c$

(c) Effect of mean stress is to change the value of $\Delta\sigma_{TH}$ and $\Delta\sigma_c$, but not to change the mid portion of the S-N curve.

~~(d) A proof test with is performed to a stress level of σ_p.~~

~~If $\sigma_p\sqrt{\pi a_0} > K_{IC}$, then the specimen fails immediately.~~

(d) Humidity increases ΔK_{TH} and thereby increases $\Delta\sigma_{TH}$.