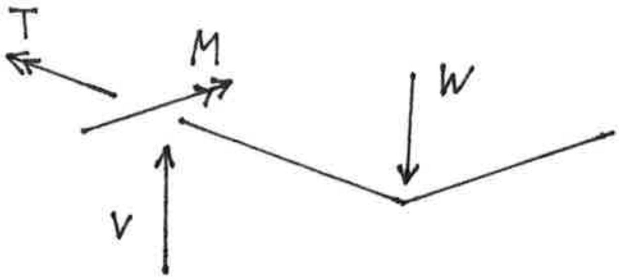
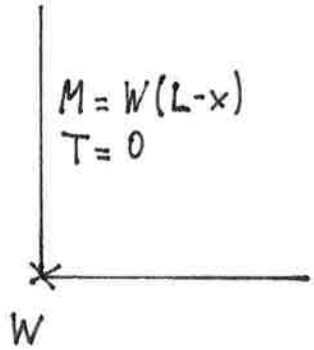
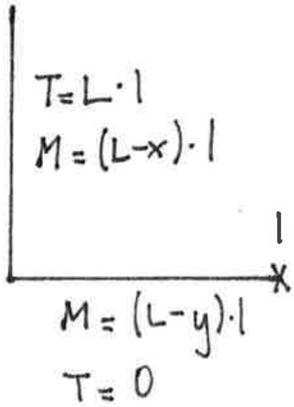
1a)



i) The non-zero boundary reactions at the built-in support are V, M and T.

ii) The degree of redundancy is one. We can, for example, remove the roller support.





Tip deflection (Principle of virtual work)

$$\delta_{10} = \frac{1}{EI}\int_0^L W(L-x)^2 dx = \frac{1}{3EI} WL^3$$

Tip deflection due to unit load at tip.

$$\delta_{11} = \frac{1}{EI}\int_0^L (L-x)^2\,dx + \frac{1}{EI}\int_0^L (L-y)^2\,dy + \frac{1}{GJ}\int_0^L L^2\,dx$$

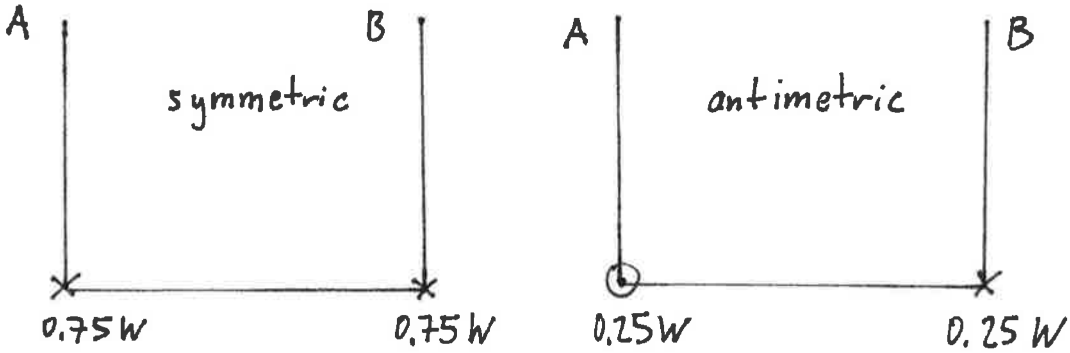
$$= \frac{1}{EI}\,\frac{1}{3}L^3 + \frac{1}{EI}\,\frac{1}{3}L^3 + \frac{1}{GJ}L^3$$

$$= \frac{2}{3}\,\frac{L^3}{EI} + \frac{L^3}{GJ} = \frac{7}{6}\,\frac{L^3}{EI}$$

$$\Rightarrow X = \frac{1}{3}WL^3 \cdot \frac{6}{7L^3} = \frac{2}{7}W$$

$$\Rightarrow M = WL - \frac{2}{7}WL = \underline{\frac{5}{7}WL}$$

1b) The loading can be decomposed in a symmetric and antimetric component.



Moments for symmetric loading

$$M_A = M_B = 0.75\,LW$$

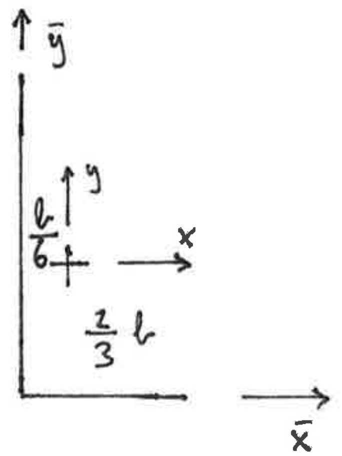
Moments for antimetric loading like in a)

$$\Rightarrow M_A = 0.75\,LW - \frac{5}{7}\,WL\,\frac{1}{4} = \underline{0.5714\,W}$$

$$\Rightarrow M_B = 0.75\,LW + \frac{5}{7}\,WL\,\frac{1}{4} = \underline{0.9286\,W}$$

2a) i) $J = \sum \frac{1}{3}\, l\, t^3 = \frac{1}{3}\, 2l\, t^3 + \frac{1}{3}\, l\, t^3 = \underline{l\, t^3}$

ii)



thin-walled cross-section

Centre of gravity:

$$3\,l\,t\,y_s = 2\,l\,t\,l \quad \Rightarrow \quad y_s = \frac{2}{3}\,l$$

$$3\,l\,t\,x_s = l\,t\,\frac{l}{2} \quad \Rightarrow \quad x_s = \frac{l}{6}$$

$$I_{xx} = \frac{(2l)^3 t}{12} + 2lt\left(l - \frac{2}{3}l\right)^2 + lt\left(\frac{2}{3}l\right)^2$$

$$= l^3 t\left(\frac{2}{3} + \frac{2}{9} + \frac{4}{9}\right) = 1{,}333\ l^3 t$$

$$I_{yy} = 2lt\left(\frac{l}{6}\right)^2 + \frac{l^3 t}{12} + lt\left(\frac{l}{2} - \frac{l}{6}\right)^2$$

$$= l^3 t\left(\frac{1}{18} + \frac{1}{12} + \frac{1}{9}\right) = 0{,}25\, l^3 t$$

$$I_{xy} = 2lt\left(-\frac{l}{6}\right)\left(\frac{1}{3}l\right) + lt\left(-\frac{2}{3}l\right)\left(\frac{1}{3}l\right)$$

$$= -0{,}333\ l^3 t$$

Centre of Mohr's circle

$$\frac{1}{2}\left(1{,}333\, l^3 t + 0{,}25\, l^3 t\right) = 0.7917\ l^3 t$$

Radius of Mohr's circle

$$\underline{l^3 t}\ \sqrt{0.1111 + (1{,}333 - 0{,}7917)^2} = 0{,}6357\ l^3 t$$

$$\Longrightarrow \quad I_{\xi\xi} = \underline{1.427\, l^3 t} \qquad I_{\eta\eta} = \underline{0.156\, l^3 t}$$

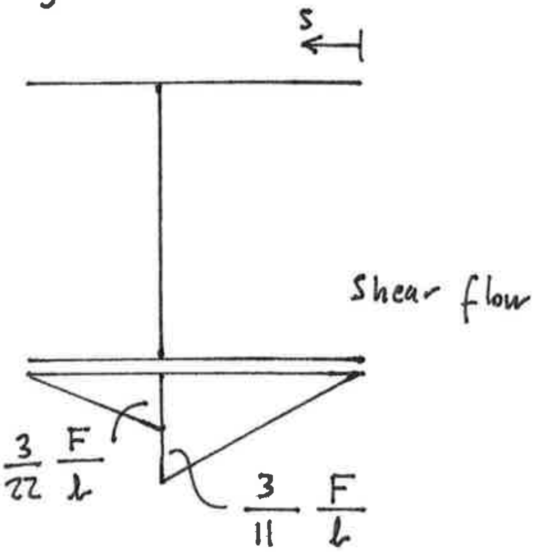
2l)

$$q = \frac{FS}{I}$$

$$I = 2\cdot\frac{3}{2} l^3 t + (2l)^3 t \frac{1}{12} = \frac{11}{3} l^3 t$$

$$S = s\cdot t\cdot l$$

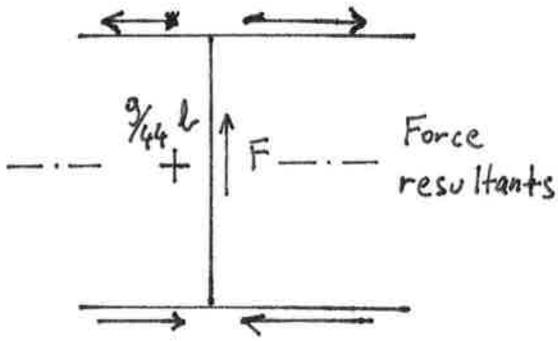
$$q = \frac{F s t l 3}{11 l^3 t}$$



Moment equilibrium

$$\left(\frac{3}{22}F - \frac{3}{88}F\right).2l = F x_s$$

$$\Longrightarrow \quad \underline{x_s = \frac{9}{44} l}$$



3a)

$$\Pi(w) = \frac{1}{2}\int_0^L \left(EI\, w''^2 - P w'^2 + k w^2\right) dx$$

$$w = \sum a_n \sin\frac{n\pi x}{L}$$

$$w' = \sum a_n \frac{n\pi}{L} \cos\frac{n\pi x}{L}$$

$$w'' = -\sum a_n \left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L}$$

$$\frac{1}{2}\int_0^L EI\, w'' dx = \frac{1}{2} EI \sum_n \sum_m a_n a_m \left(\frac{n\pi}{L}\right)^2 \left(\frac{m\pi}{L}\right)^2 \int_0^L \sin\frac{n\pi x}{L} \sin\frac{n\pi x}{L}\, dx$$

with $\displaystyle\int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = \begin{cases} L/2 & n=m \\ 0 & n \neq m \end{cases}$

First term of $\Pi(w)$ $\quad \frac{1}{2} EI \left(\frac{L}{2}\right) \sum a_n^2 \left(\frac{n\pi}{L}\right)^4$

Second term of $\Pi(w)$ $\quad -\frac{1}{2} P \left(\frac{L}{2}\right) \sum a_n^2 \left(\frac{n\pi}{L}\right)^2$

Third term of $\Pi(w)$ $\quad \frac{1}{2} k \left(\frac{L}{2}\right) \sum a_n^2$

We can write for $\Pi(w)$

$$\Pi(a_n) = \frac{1}{2}\, \underline{a}^T \underline{\underline{K}}\, \underline{a}$$

with $K_{nn} = \frac{L}{2}\left[\left(\frac{n\pi}{L}\right)^4 EI - \left(\frac{n\pi}{L}\right)^2 P + k\right]$

Instability when $K_{nn} = 0$

$$\Rightarrow \quad P_{cr} = \left(\frac{L}{n\pi}\right)^2 \left[\left(\frac{n\pi}{L}\right)^4 EI + k \right]$$

$$= \left(\frac{n\pi}{L}\right)^2 EI + \left(\frac{L}{n\pi}\right)^2 k$$

For given n, the length that minimises P_{cr}

$$\frac{dP_{cr}}{dL} = 0 \quad \Rightarrow -\frac{2}{L^3} n^2 \pi^2 EI + \frac{2L}{n^2\pi^2} k = 0$$

whence

$$L_{cr}^4 = \frac{n^4 \pi^4 EI}{k} \quad \Rightarrow L_{cr} = n\pi \left(\frac{EI}{k}\right)^{\frac{1}{4}}$$

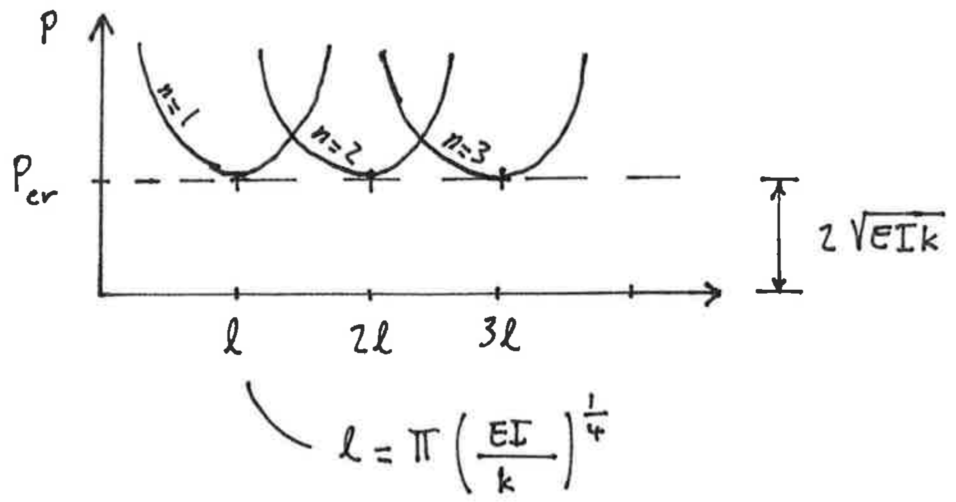
The minimum value of P_{cr} at this length L_{cr}

$$= \left(\frac{n\pi}{L_{cr}}\right)^2 EI + \left(\frac{L_{cr}}{n\pi}\right)^2 k$$

$$= \frac{n^2\pi^2}{n^2\pi^2} \frac{\sqrt{k}}{\sqrt{EI}} EI + \frac{n^2\pi^2}{n^2\pi^2} \frac{\sqrt{EI}}{\sqrt{k}} k$$

$$= \sqrt{k}\sqrt{EI} + \sqrt{k}\sqrt{EI}$$

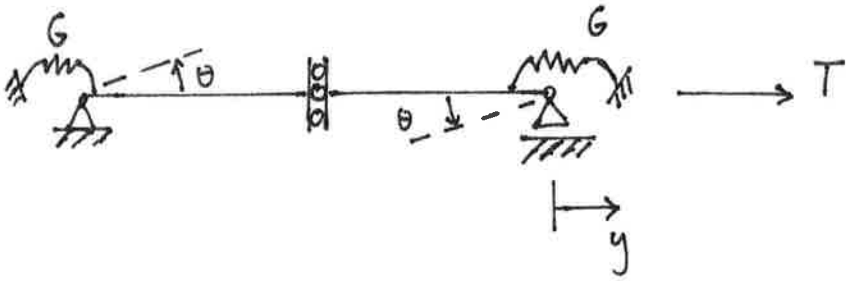
$$\Rightarrow \quad P_{cr\,min} = 2\sqrt{EIk}$$



3b) 1) The analysis assumes only radially symmetric modes, but modes which also have circumferential wave numbers - e.g. checkerboard modes - might be relevant. Such modes were seen in the 3D4 buckling lab.

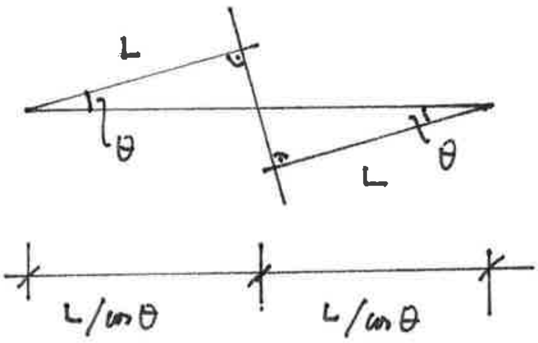
2) Thin-walled cylinders are <u>very</u> sensitive to imperfections thus any theoretical prediction using 'perfect' elastic stability theory is likely to be significantly greater than the actual load-carrying capacity of the real cylinder. Again, this was observed in the 3D4 buckling lab.

4a)



$$\text{Total potential energy} = \underbrace{\left(\tfrac{1}{2} G\theta^2\right)\cdot 2}_{\text{internal}} - \underbrace{Ty}_{\text{external}}$$

Deflected shape



Extension $\quad y = 2\left(\dfrac{L}{\cos\theta} - L\right)$

Small angles $\quad \cos\theta = 1 - \dfrac{\theta^2}{2} + \dots$

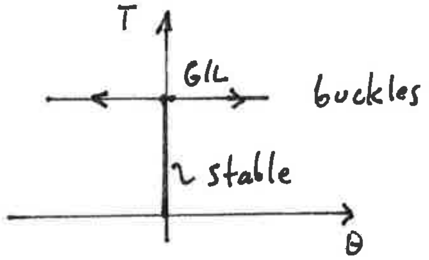
Binomial theorem $\quad \dfrac{1}{\cos\theta} = \left(1 - \dfrac{\theta^2}{2} + \dots\right)^{-1} = 1 + \dfrac{\theta^2}{2} + \dots$

$$\text{Total P.E.} = G\theta^2 - 2T\left(L\left(1 + \frac{\theta^2}{2} + \dots\right) - L\right)$$

$$= G\theta^2 - TL\theta^2$$

Equilibrium from $\frac{\partial \Pi(\theta)}{\partial \theta} = 2\theta(G - TL) = 0$

Solutions $\theta = 0$ unless $G - TL = 0 \Longrightarrow T = \frac{G}{L}$ critical tension



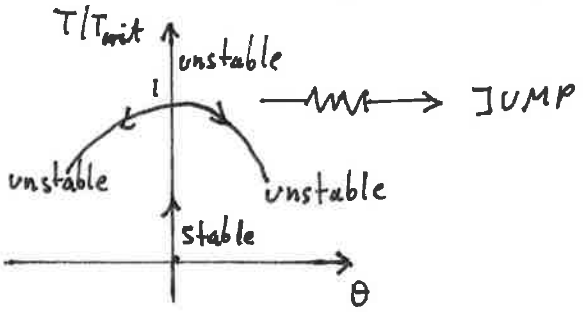
To determine bifurcation type, need large angle theory

$$\Pi(\theta) = G\theta^2 - 2TL\left(\frac{1}{\cos\theta} - 1\right)$$

Equilibrium from $\frac{\partial \Pi}{\partial \theta} = 2\theta G - 2TL\left(\frac{\sin\theta}{\cos^2\theta}\right) = 0$

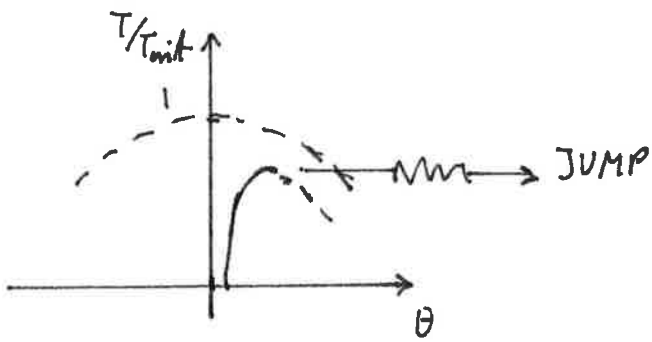
Solutions $\theta = 0$ unless $\frac{T}{T_{crit}} = \frac{\theta\cos^2\theta}{\sin\theta}$

Now just plot it by evaluating few points



Unstable symmetric bifurcation

And with imperfections



4b)

Shanley's resolution to the Column Paradox involved the notion that a structure would undergo a deformation $d\underline{w}$ during a loading increment $d\underline{F}$. This differs from basic elastic stability theory which tends to start from the premise that the structure is carrying a fixed load $\underline{F}$ and then virtual (or small) displacements $d\underline{w}$ are explored about the equilibria to determine stability. It is Shanley's joint combination of $d\underline{w}$ and $d\underline{F}$ that is the crucial difference. Looking at $d\underline{w}$ alone does not find all possible solutions in case of inelastic buckling.

4c)

$$\begin{bmatrix} M_B \\ M_C \end{bmatrix} = \begin{bmatrix} K_1 & K_2 \\ K_2 & K_3 \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix}$$

Beam AB provides $\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \frac{EI}{L}$

BC provides $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \frac{EI}{L}$

CD provides $\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \frac{EI}{L}$

Column BE provides $\begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix} \frac{EI}{L}$

Column CF:

$$\begin{bmatrix} M_C \\ M_F \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} s & sc \\ sc & s \end{bmatrix} \begin{bmatrix} \theta_C \\ \theta_F \end{bmatrix}$$

but $M_F = 0$

$$0 = \frac{EI}{L} \left(sc\,\theta_C + s\,\theta_F \right)$$

$$\Longrightarrow \quad \theta_F = -c\,\theta_C$$

Substituting into first equation yields

$$M_C = \frac{EI}{L} \left[s\,\theta_C + sc\left(-c\,\theta_C\right) \right]$$

$$= \frac{EI}{L} s \left(1 - c^2\right) \theta_C$$

$\Longrightarrow$

Total stiffness matrix $\quad \frac{EI}{L} \begin{bmatrix} 8+s & 2 \\ 2 & 7 + s(1-c^2) \end{bmatrix}$

Numerical answers for 3D4

1a) ii) $M = \frac{5}{7} WL$

1b) $M_A = 0.5714\, W$

$M_B = 0.9286\, W$

2a) i) $J = b t^3$

ii) $I_{\xi\xi} = 1.427\, b^3 t$

$I_{\eta\eta} = 0.156\, b^3 t$

2b) $x_s = \frac{9}{44} b$

3a) $P_{cr,min} = 2\sqrt{EIk}$

4a) $T_{min} = \frac{G}{L}$

4c) $\begin{bmatrix} 8+s & 2 \\ 2 & 7+s(1-c^2) \end{bmatrix} \frac{EI}{L}$