3D7 2012 1.

1 a) Multiply all terms by a weight function w and integrate by parts. Considering LHS,

$$\int_0^L w \frac{d^2}{dx^2}\left(EI \frac{d^2 v}{dx^2}\right) dx + \int_0^L w \alpha v \, dx$$

$$= -\int_0^L \frac{dw}{dx} \frac{d}{dx}\left(EI \frac{d^2 v}{dx^2}\right) dx + w \frac{d}{dx}\left(EI \frac{d^2 v}{dx^2}\right)\Big|_0^L$$

$$+ \int_0^L w \alpha v \, dx$$

Integrate by parts again,

$$= \int \frac{d^2 w}{dx^2} EI \frac{d^2 v}{dx^2} - \frac{dw}{dx} EI \frac{d^2 v}{dx^2}\Big|_0^L + w \frac{d}{dx}\left(EI \frac{d^2 v}{dx^2}\right)\Big|_0^L$$

$$+ \int_0^L w \alpha v \, dx$$

Boundary condition for beam in question:

$$\cancel{} \; v = \frac{dv}{dx} = 0 \quad @ \quad x = 0$$

(zero deflection & zero rotation)

$$\Longrightarrow \quad w = \frac{dw}{dx} = 0 \quad @ \quad x = 0$$

$$EI \frac{d^2 v}{dx^2} = \frac{d}{dx}\left(EI \frac{d^2 v}{dx^2}\right) = 0 \quad @ \quad x = L$$

(zero moment & zero shear force)

Therefore, suitable weak form is:

$$\int_0^L \frac{d^2 w}{dx^2} EI \frac{d^2 v}{dx^2} dx + \int_0^L w \alpha v \, dx = \int_0^L w f \, dx$$

1 cont b) Must have continuous first derivatives in order to compute second derivatives in weak form.

Insufficient continuity → unstable response

c) Length $\ell = x_2 - x_1$

For $\alpha = 0$

$$k_{11} = \int_0^\ell EI \frac{d^2N_1}{dx^2} \frac{d^2N_1}{dx^2}\, dx$$

where N_1 is a Hermite (C^1) shape function. From Data Sheet

$$N_1 = (x-\ell)^2(\ell+2x)/\ell^3$$

$$\frac{dN_1}{dx} = \left(2(x-\ell)(\ell+2x) + 2(x-\ell)^2\right)/\ell^3$$

$$\frac{d^2N_1}{dx^2} = (12x - 6\ell)\ell^3$$

$$k_{11} = \int_0^\ell EI\left[(12x-6\ell)/\ell^3\right]^2 dx$$

$$= \frac{EI}{\ell^6}\int_0^\ell 144x^2 - 72\ell x + 36\ell^2\, dx$$

$$= \frac{48EI}{\ell^3}$$

Question ~~1~~ 2

a) $N_1 = 1 - \frac{x}{4.0} - \frac{y}{2.0}$

$N_2 = \frac{x}{4.0}$

$N_3 = \frac{y}{2.0}$

b) $u_x = N_1 u_{x1} + N_2 u_{x2} + N_3 u_{x3}$

$u_y = N_1 u_{y1} + N_2 u_{y2} + N_3 u_{y3}$

Unconstrained dofs are u_{x2} and u_{y2}

$$\varepsilon_{xx} = \frac{\partial N_2}{\partial x} u_{x2} = \frac{u_{x2}}{4.0}$$

$$\varepsilon_{yy} = \frac{\partial N_2}{\partial y} u_{y2} = 0$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial N_2}{\partial y} u_{x2} + \frac{\partial N_2}{\partial x} u_{y2}\right) = \frac{1}{8.0} u_{y2}$$

c) Plane stress

$$D = E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

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$$B = \begin{bmatrix} \frac{\partial N_2}{\partial x} & 0 \\ 0 & \frac{\partial N_2}{\partial y} \\ \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

$$K = B^T D B = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & \frac{E}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \cdot 4.0$$

$$= E \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{8} \end{bmatrix}$$

d)

$$M = \begin{bmatrix} \int N_2 N_2 & 0 \\ 0 & \int N_2 N_2 \end{bmatrix}$$

$$\int N_2 N_2 \, d\Omega = \int_0^4 \int_0^{2-\frac{x}{2}} \frac{x^2}{16} \, dy\, dx = \int_0^4 \frac{x^2}{16}\left(-\frac{x}{2} + 2\right) dx$$

$$= \int_0^4 \left(-\frac{x^3}{32} + \frac{x^2}{8}\right) dx = \left[-\frac{x^4}{128} + \frac{x^3}{24}\right]_0^4$$

$$= \frac{2}{3}$$

2a) Multiply with test function w_0 and integrate

$$k\int\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)w_0 + c_x\int\frac{\partial T}{\partial x}w_0 + c_y\int\frac{\partial T}{\partial y}w_0 + \int s w_0 = 0$$

$$\underbrace{k\int_\Omega \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}w_0\right)}_{k\int_\Gamma \frac{\partial T}{\partial x}w_0 n_x = 0} = k\int_\Omega \frac{\partial^2 T}{\partial x^2}w_0 + k\int_\Omega \frac{\partial T}{\partial x}\frac{\partial w_0}{\partial x}$$

$$\Rightarrow k\int_\Omega \frac{\partial^2 T}{\partial x^2}w_0 = -k\int_\Omega \frac{\partial T}{\partial x}\frac{\partial w_0}{\partial x}$$

$$k\int_\Omega\left(\frac{\partial w_0}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial w_0}{\partial y}\frac{\partial T}{\partial y}\right) - c_x\int_\Omega w_0\frac{\partial T}{\partial x}\,d\Omega - c_y\int w_0\frac{\partial T}{\partial y} = \int_\Gamma w_0 s\, d\Omega$$

$$\int_{-5}^{5}\int_{-5}^{5} N_1 \frac{\partial N_2}{\partial x}\, dy\, dx = \frac{1}{80}\int_{-5}^{5}\left[y - \frac{2y^2}{10} + \frac{y_3}{75} - \frac{xy^2}{5} + \frac{xy^3}{75}\right]_{-5}^{5}$$

$$= \frac{1}{80}\int_{-5}^{5}\left[\frac{40}{3} + \frac{10x}{3}\, dx\right] = \frac{5}{3}$$

$$N_1 \frac{\partial N_2}{\partial y} = -\frac{1}{80}\left(1 - \frac{x}{5}\right)\left(1 - \frac{y}{5}\right)\left(1 + \frac{x}{5}\right)$$

$$= -\frac{1}{80}\left(1 + \frac{x^2 y}{125} - \frac{y}{5} - \frac{x^2}{25}\right)$$

$$\int N_1 \frac{\partial N_2}{\partial y} = -\frac{1}{80}\int\left[y + \frac{x^2y^2}{250} - \frac{y^2}{10} - \frac{x^2 y}{25}\right]_{-5}^{5} dy$$

$$= -\frac{1}{80}\int\left[10 - \frac{x^2}{25}\, dx = -\frac{1}{80}\left[10x - \frac{2x^3}{15}\right]_{-5}^{5}\right.$$

$$= -\frac{5}{6}$$

$$C_{12} = \frac{5}{3} c_x - \frac{5}{6} c_y$$

3,2c) For linear shape functions:

The maximum polynomial order in integrand is cubic

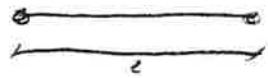
$$(\underset{\substack{\uparrow \\ \text{quadratic}}}{N} \quad \underset{\substack{\uparrow \\ \text{linear}}}{\frac{\partial N}{\partial x}} \quad \underset{\substack{\uparrow \\ \text{linear}}}{\frac{\partial N}{\partial y}})$$

$2n-1$

for $n=2$ we can integrate cubic polynomials accurately

$\Rightarrow$ 2×2 integration points

4 a) 1D element



$$\underline{\underline{k}}_e = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{(Stiffness matrix)}$$

$$\underline{\underline{m}}_e = \rho l \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \quad \text{(consistent mass matrix)}$$

Generalized eigenvalue problem:

$$\det(\underline{\underline{k}}_e - \lambda \underline{\underline{m}}_e) = 0$$

$$\begin{vmatrix} EA/l - \lambda\rho l/3 & -EA/l - \lambda\rho l/6 \\ -EA/l - \lambda\rho l/6 & EA/l - \lambda\rho l/3 \end{vmatrix} = 0$$

$$= (EA/l - \lambda\rho l/3)^2 - (-EA/l - \lambda\rho l/6)^2 = 0$$

$$= \cancel{\frac{(EA)^2}{l^2}} - \frac{2EA\lambda\rho l}{l\,3} + \frac{\lambda^2\rho^2 l^2}{9}$$

$$- \frac{(EA)^2}{l^2} - \frac{2EA\lambda\rho l}{l\,6} - \frac{\lambda^2\rho^2 l^2}{36} = 0$$

$$= \left(\rho^2 l^2 \frac{3}{36}\right)\lambda^2 - (EA\rho)\lambda = 0$$

$$\lambda = \frac{EA\rho}{\rho^2 l^2 (3/36)} = \frac{12EA}{\rho l^2}$$

$$\lambda^{max} \sim 1/l^2$$

4 b) Local problem gives upper bound on λ_{max}, and will therefore yield a conservative estimate for the critical time step.

c) $$y_{n+1} = A y_n$$

Forward Euler:

$$y_{n+1} = y_n + \Delta t\, f(y_n, t_n)$$

For $\dot{y} + \lambda y = 0 \rightarrow \dot{y} = -\lambda y$

$$\therefore\ y_{n+1} = y_n - \cancel{\Delta t y_n}\ \Delta t \lambda y_n$$

$$\cancel{= (1 - \Delta t y_n)}$$

$$= \underbrace{(1 - \Delta t \lambda)}_{A} y_n$$

For $|A| \leq 1 \rightarrow |(1 - \Delta t \lambda)| \leq 1$

$$\Longrightarrow \Delta t \leq \frac{2}{\lambda}$$

Largest eigenvalue determines critical time step. Since

$$\lambda^{max} \sim 1/e^2$$

$$\Delta t_{crit} \propto e^2$$

$\rightarrow$ critical time step decreases rapidly with mesh refinement, hence of limited practical use.