ENGINEERING PART IIA 3F1 – SIGNALS AND SYSTEMS – May 2012
SOLUTIONS

1 (a) Taking the z–transform of both sides of the difference equation with zero initial conditions, and putting all $Y(z)$ terms on the left, gives $Y(z)(z^2 - z + \alpha) = X(z)$. Rearranging yields

$$\frac{Y(z)}{X(z)} = \frac{1}{z^2 - z + \alpha}$$

[10%]

(b) $\alpha = 0$: Poles are the roots of $z^2 - z = 0$, which gives $z = 0$, $z = 1$. Hence, the system is (marginally) unstable as one pole is on the unit circle, whereas both must be inside it for true stability.

$\alpha = \frac{1}{4}$: Poles are the roots of $z^2 - z + \frac{1}{4} = 0$, which gives two poles at $z = \frac{1}{2}$. Hence, the system is stable (and has critical damping).

$\alpha = \frac{1}{2}$: Poles are the roots of $z^2 - z + \frac{1}{2} = 0$, which gives $z = \frac{1 \pm i}{2}$. Since the magnitude of the poles is $1/\sqrt{2}$, which is less than one, the system is stable.

$\alpha = 2$: Poles are the roots of $z^2 - z + 2 = 0$, which gives $z = \frac{1 \pm i\sqrt{7}}{2}$. Since the magnitude of the poles is $\sqrt{2}$, which is larger than one, the system is unstable. [20%]

(c) Since x_k is a pulse, $X(z) = 1$. Then for

$\alpha = 0$:

$$Y(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}.$$

Hence, $y_k = \{0, -1, 0, 0, 0, \cdots\} + \{0, 1, 1, 1, 1, \cdots\} = \{0, 0, 1, 1, 1, \cdots\}$.

$\alpha = 1/4$:

$$Y(z) = \frac{1}{(z - \frac{1}{2})^2} = 2z^{-1} \frac{\frac{1}{2}z^{-1}}{(1 - \frac{1}{2}z^{-1})^2}.$$

Hence, using the hint with a 1-sample delay, $y_k = 0$ for $k = 0, 1$ and $y_k = 2(k-1)(1/2)^{k-1}$ for $k \geq 2$.

$\alpha = 1/2$:

$$Y(z) = \frac{1}{z^2 - z + \frac{1}{2}} = \frac{z^{-2}}{1 - 2\frac{1}{\sqrt{2}}\cos\frac{\pi}{4} z^{-1} + \left(\frac{1}{\sqrt{2}}\right)^2 z^{-2}},$$

Hence, using the data book result with a delay of 2 samples due to the numerator term, $y_k = 0$ for $k = 0, 1$ and $y_k = \left(\frac{1}{\sqrt{2}}\right)^{k-3} \sin\left(\frac{\pi}{4}(k-1)\right)$ for $k \geq 2$.

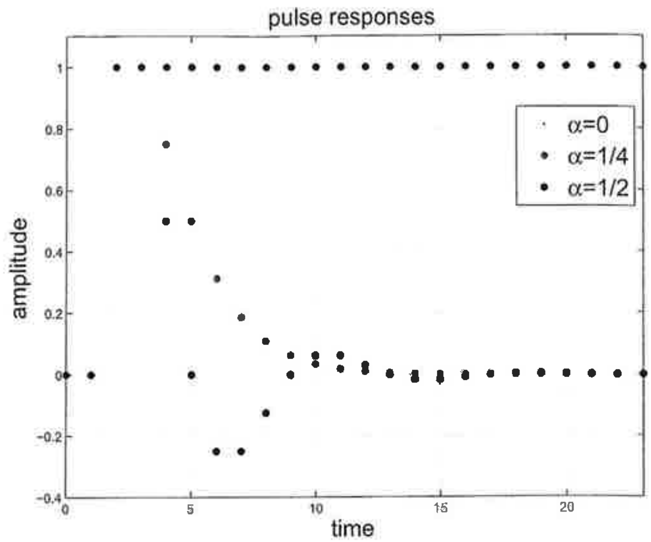

Fig. 1

The pulse responses can be seen in Fig. 1. Note that all three responses are the same up to $k = 3$. [30%]

(d) (i) Denote the transfer function from X to Y, evaluated above, as G; i.e. $Y = GX$. Since $X = K(R-Y)$ then $Y = GK(R-Y)$ and so $Y(GK+1) = GKR$. Hence,

$$\frac{Y}{R} = \frac{GK}{GK+1} = \frac{K}{z^2 - z + \alpha + K}$$

[15%]

(ii) The denominator of this expression determines stability, and it is the same as that in part (a), except that α is replaced by $\alpha + K$. Hence, using the results from part (b), the system will be stable if $0 < \alpha + K < 1$. Critical damping is achieved if $\alpha + K = \frac{1}{4}$, and stability will be achieved if $-\alpha < K < 1 - \alpha$. Thus there is no single K which will stabilise the system if α varies from 0 to 2.

For critical damping, suitable values of K, for the four given values of α, would be $\frac{1}{4}, 0, -\frac{1}{4}, -\frac{7}{4}$, although $K = 0$ would not be practically useful since the numerator of the closed-loop transfer function would then become zero. Hence we would need to deviate from critical damping when $\alpha = \frac{1}{4}$. [25%]

2 (a) Discrete-time system stability:

(i) A discrete-time system is defined to be stable if bounded inputs always produce bounded outputs. Consequences of this are that the sum of the magnitudes of the system impulse response coefficients should be finite and that all poles of the system z-transfer function should lie within the unit circle (but these are not *definitions* of stability). [10%]

(ii) For the given system, consider a step input $\{u_k\} = \{1,1,1,1,\cdots\}$ which *matches* the impulse response of the system. This is a bounded input. Taking its $z-$transform gives

$$U(z) = \frac{1}{1-z^{-1}}.$$

Hence,

$$Y(z) = G(z)U(z) = \frac{z^{-1}}{(1-z^{-1})^2}$$

Using the hint, $y_k = k$ for all $k \geq 0$, which is not a bounded output. Therefore, the system is unstable. [30%]

(b) Sums of random variables, $Y = X_1 + X_2$:

(i) Hence $X_2 = Y - X_1$.
If $X_1 = x_1$, then $X_2 = Y - x_1$, where x_1 is a constant.
Now $f(y|x_1)$ is the pdf of Y at the point y, given that $X_1 = x_1$. Hence this is the pdf of X_2 at the point $(y - x_1)$.
Hence $f(y|x_1) = f_2(y-x_1)$.
Using the marginal probability integral formula:

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f(y|x_1)\, f_1(x_1)\, dx_1 \\ &= \int_{-\infty}^{\infty} f_2(y-x_1)\, f_1(x_1)\, dx_1 \\ &= f_2(x_2) \text{ convolved with } f_1(x_1) \end{aligned}$$

[20%]

(ii) $\Phi_X(u) = E[e^{juX}] = \int_{-\infty}^{\infty} e^{jux}\, f_X(x)\, dx = \mathscr{F}(-u)$
where $\mathscr{F}(u)$ is the Fourier transform of $f_X(x)$.
Since $f_Y(y) = f_2(x_2)$ convolved with $f_1(x_1)$

$$\Phi_Y(-u) = \Phi_{X_2}(-u) \, . \, \Phi_{X_1}(-u)$$

Replacing $-u$ by u gives $\Phi_Y(u) = \Phi_{X_2}(u) \, . \, \Phi_{X_1}(u)$ [20%]

(iii) Whenever independent random processes are combined together in a linear way, their pdfs are convolved together. Convolution is a tricky operator to use, so it is easier to work with characteristic functions of pdfs where the Φ's are multiplied together instead, and we work in the 'u' (negative frequency) domain.

Characteristic functions are also useful for calculating n^{th} order moments of a given pdf by taking the n^{th} derivative of its $\Phi(u)$ and then setting $u = 0$. [20%]

3 (a) The PSD of an ergodic random signal is the Fourier transform of the ACF of the signal. [10%]

(b) $r_{XX}(\tau)$ is a triangular pulse of width $2T_0$, height P, centred on zero.

From the E&I Data Book (p 21), the Fourier transform of this ACF is

$$S_X(\omega) = P\,T_0\,\text{sinc}^2\left(\frac{\omega T_0}{2}\right) = \frac{2P}{T_0}\left(\frac{1-\cos(\omega T_0)}{\omega^2}\right)$$

(The second version is obtained from a direct FT of the triangular waveform.) [20%]

(c) For the linear system, $Y(\omega) = H(\omega)\,X(\omega)$. Therefore, in terms of power:

$$|Y(\omega)|^2 = |H(\omega)|^2\,|X(\omega)|^2$$

Hence the output PSD is given by

$$S_Y(\omega) = |Y(\omega)|^2 = |H(\omega)|^2\,S_X(\omega) = |H(\omega)|^2\,P\,T_0\,\text{sinc}^2\left(\frac{\omega T_0}{2}\right)$$

Alternatively this can be derived by convolution in the time domain and then by taking the Fourier transform of the result as done in the lecture notes. [20%]

(d) From $r_{XX}(\tau)$ and $r_{YY}(\tau)$ we may use the Fourier transform to compute

$$S_X(\omega) = \text{FT}\{r_{XX}(\tau)\} \quad \text{and} \quad S_Y(\omega) = \text{FT}\{r_{YY}(\tau)\}$$

Hence

$$|H(\omega)|^2 = \frac{S_Y(\omega)}{S_X(\omega)} \quad \text{and so} \quad |H(\omega)| = \sqrt{\frac{S_Y(\omega)}{S_X(\omega)}}$$

Thus we may easily compute the *magnitude* of $H(\omega)$ but not its *phase*.

We must also ensure that $r_{XX}(\tau)$ and $r_{YY}(\tau)$ are measured over a sufficiently long time interval that statistical fluctuations due to the random nature of X are minimised. [30%]

(e) It can be shown that $r_{XY}(\tau) = r_{XX}(\tau) * h(\tau)$ (convolution)

and so $S_{XY}(\omega) = S_X(\omega)\,H(\omega)$ where $S_{XY}(\omega) = FT\{r_{XY}(\tau)\}$.

Hence

$$H(\omega) = \frac{S_{XY}(\omega)}{S_X(\omega)}$$

and both the *amplitude and phase* of $H(\omega)$ may be obtained. [20%]

4 (a) For a prefix-free binary code to exist, the Kraft inequality must be satisfied:

$$\sum_{i=1}^{N} 2^{-l_i} \leq 1$$

for N codewords of lengths l_i, where $i = 1 \ldots N$.

In the case of a Huffman code, this becomes an equality to make the code as efficient as possible.

Code (i): $\sum_{i=1}^{N} 2^{-l_i} = 0.5 + 0.25 + 0.25 + 0.125 = 1.125$

Hence no code exists.

Code (ii): $\sum_{i=1}^{N} 2^{-l_i} = 0.25 + 0.25 + 0.25 + 0.125 = 0.875$

Hence a valid code exists but it could not be a Huffman code because the result is not 1.

Code (iii): $\sum_{i=1}^{N} 2^{-l_i} = 0.5 + 0.25 + 0.125 + 0.125 = 1.0$

Hence a valid code exists and it could be a Huffman code. [30%]

(b) (i) $P_X(0) = p$ and so $P_X(1) = 1 - p$

Therefore $H(X) = h(p)$, using the binary entropy function with $b = 2$ for h to be in units of 'bits'.

$P_{Y|X}(1|1) = 1$ and so $P_{Y|X}(0|1) = 0$ because pmfs must sum to unity.

Similarly $P_{Y|X}(1|0) = \frac{1}{2}$ and so $P_{Y|X}(0|0) = \frac{1}{2}$.

Therefore

$$H(Y|X) = h(1)\, P_X(1) + h(\tfrac{1}{2})\, P_X(0) = 0\,.\,(1-p) + 1\,.\,p = p$$

since $h(1) = 0$ and $h(\frac{1}{2}) = 1$ if $b = 2$.

The joint probability matrix for $(X\ Y)$ is $\begin{bmatrix} p/2 & 0 \\ p/2 & 1-p \end{bmatrix}$ where the two columns correspond to $X = 0$ and $X = 1$, and the two rows to $Y = 0$ and $Y = 1$ respectively. Therefore

$$\begin{aligned} H(XY) &= -2\,\tfrac{p}{2}\,\log_2(\tfrac{p}{2}) - (1-p)\,\log_2(1-p) \\ &= -p\,\log_2(p) + p\,\log_2(2) - (1-p)\,\log_2(1-p) \\ &= h(p) + p \end{aligned}$$

Summing along the rows of the joint probability matrix above gives

$$P_Y(0) = \tfrac{p}{2} + 0 = \tfrac{p}{2} \quad \text{and} \quad P_Y(1) = \tfrac{p}{2} + (1-p) = 1 - \tfrac{p}{2}$$

Therefore $H(Y) = h(\frac{p}{2})$.

$$\begin{aligned} H(X|Y) &= h(0)\,.\,P_Y(0) + h\left(\tfrac{p/2}{1-p/2}\right).\,P_Y(1) \\ &= 0 + h\left(\tfrac{p}{2-p}\right).\left(1-\tfrac{p}{2}\right) \end{aligned}$$

Alternatively we may obtain a simpler looking result using

$$H(X|Y) = H(XY) - H(Y) = h(p) + p - h(\tfrac{p}{2})$$

Finally

$$I(X;Y) = H(X) - H(X|Y) = h(p) - \left(h(p) + p - h(\tfrac{p}{2})\right) = h(\tfrac{p}{2}) - p$$

As a check, we can get the same result from $I(Y;X) = H(Y) - H(Y|X)$. [30%]

(ii) The channel is described by $P_Y = P(Y|X)\,P_X$ where

$$P(Y|X) = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$$

and P_X and P_Y are column vectors of event probabilities for X and Y. The matrix elements are obtained from $P(0|0), P(0|1), P(1|0), P(1|1)$ calculated above. [15%]

(iii) The capacity of the channel is the maximum value of $I(X;Y)$ over all choices of P_X, and hence over all choices of p. Hence at the maximum:

$$\frac{dI(X;Y)}{dp} = h'(\tfrac{p}{2})\,.\,\tfrac{1}{2} - 1 = 0$$

Using the formula given for $h'(x)$ when $x = p/2$, we get

$$\begin{aligned} \log_2\left(\frac{1-p/2}{p/2}\right) &= 2 \\ \frac{2-p}{p} &= 4 \\ 5p &= 2 \\ p &= 0.4 \end{aligned}$$

Therefore the channel capacity is given by
$I_{\max}(X;Y) = h(p/2) - p = h(0.2) - 0.4 = 0.3219$ bits per use [25%]

N G Kingsbury, June 2012.

ENGINEERING TRIPOS PART IIA
Tuesday 1 May 2012 9 to 10.30
Module 3F1 SIGNALS AND SYSTEMS
Short answers

1. (a) $\dfrac{1}{z^2 - z + \alpha}$;
   (b) $z = 0, z = 1$, unstable; $z = \frac{1}{2}, z = \frac{1}{2}$, stable;
   $z = \frac{1 \pm i}{2}$, stable; $z = \frac{1 \pm i\sqrt{7}}{2}$, unstable.
   (c) $y_k = \{0, 0, 1, 1, 1, \cdots\}$;
   $y_k = 0$ for $k = 0, 1$, $y_k = 2(1-k)(1/2)^{k-1}$ for $k \geq 2$;
   $y_k = 0$ for $k = 0, 1$, $y_k = (1/\sqrt{2})^{k-3} \sin(\frac{\pi}{4}(k-1))$ for $k \geq 2$.
   (d) (i) $\dfrac{K}{z^2 - z + \alpha + K}$;
   (ii) $0 < K < 1$; $-\frac{1}{4} < K < \frac{3}{4}$; $-\frac{1}{2} < K < \frac{1}{2}$; $-2 < K < -1$; No.

2. (b) (i) $f_Y(y) = \int_{-\infty}^{\infty} f_2(y - x_1)\, f_1(x_1)\, dx$ (convolution)
   (ii) $\Phi_Y(u) = \Phi_{X_2}(u)\, \Phi_{X_1}(u)$

3. (b) $S_X(\omega) = P\, T_0\, \text{sinc}^2\left(\dfrac{\omega T_0}{2}\right) = \dfrac{2P}{T_0}\left(\dfrac{1 - \cos(\omega T_0)}{\omega^2}\right)$
   (c) $S_Y(\omega) = |H(\omega)|^2\, P\, T_0\, \text{sinc}^2\left(\dfrac{\omega T_0}{2}\right)$
   (d) $|H(\omega)| = \sqrt{\dfrac{S_Y(\omega)}{S_X(\omega)}}$
   (e) $H(\omega) = \dfrac{S_{XY}(\omega)}{S_X(\omega)}$

4. (a) (i) no, yes, yes; (ii) Huffman only for third code.
   (b) (i) $H(X) = h(p)$; $H(Y|X) = p$; $H(XY) = h(p) + p$; $H(Y) = h(p/2)$;
   $H(X|Y) = h\left(\frac{p}{2-p}\right) \cdot (1 - \frac{p}{2}) = h(p) + p - h(p/2)$
   $I(X;Y) = h(p/2) - p$
   (iii) $p_{\max} = 0.4$; $I_{\max}(X;Y) = 0.3219$ bit per use.