3F2 Systems and Control: 2012 Solutions

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1. (a) The matrix exponential is defined, in analogy with the scalar exponential function, by the series:

$$e^M = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots \tag{1}$$

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(b) From the definition above we have

$$e^{-At} = I - At + \frac{A^2t^2}{2!} - \frac{A^3t^3}{3!} + \cdots \tag{2}$$

Hence

$$\begin{aligned} \frac{d}{dt}\left(e^{-At}\right) &= -A + A(At) - A\frac{A^2t^2}{2!} + \cdots && (3)\\ &= -A\left(I - At + \frac{A^2t^2}{2!} - \frac{A^3t^3}{3!} + \cdots\right) && (4)\\ &= -Ae^{-At} && (5) \end{aligned}$$

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Alternatively, the factor $-A$ can be 'pulled out' on the right of (3), leading to $-e^{-At}A$.

(c) Two ways of evaluating e^{At} were covered in the course:

i. If W is the matrix of eigenvalues of A, and is invertible, so that $AW = W\Lambda$, or $A = W\Lambda W^{-1}$, where Λ is the diagonal matrix with the ith eigenvalue λ_i of A as its (i,i) entry, then, from (1),

$$e^{At} = We^{\Lambda t}W^{-1} \tag{6}$$

and $e^{\Lambda t}$ is just the diagonal matrix with $e^{\lambda_i t}$ as its (i,i) entry.

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ii. From (1) it follows that the Laplace transform of e^{At} is $(sI - A)^{-1}$. So this inverse matrix can be evaluated, then the entries of e^{At} can be obtained by taking the inverse Laplace transform of each element of $(sI - A)^{-1}$.

iii. Other methods of evaluating e^{At} are acceptable answers, despite not being covered in the course. For example, just evaluating sufficiently many terms in the infinite series (1) is possible — although numerical problems arise if this is done naively.

(d) From $\dot{x} = Ax + Bu$ we have

$$\dot{x} - Ax = Bu \tag{7}$$

Premultiply both sides by e^{-At} (integrating factor):

$$e^{-At}\dot{x} - e^{-At}Ax = e^{-At}Bu \tag{8}$$

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Hence

$$\frac{d}{dt}\left(e^{-At}x\right) = e^{-At}Bu \tag{9}$$

or from $X(s) = (sI-A)^{-1}x(0) + (sI-A)^{-1}BU(s)$, using Databook & L.T. tables. but don't assume $e^{At} \leftrightarrow (sI-A)^{-1}$

Integrating both sides from 0 to t gives

$$e^{-At}x(t) - x(0) = \int_0^t e^{-A\tau}Bu(\tau)d\tau \tag{10}$$ 2

Multiplying both sides by e^{At} and re-arranging gives

$$x(t) = e^{At}x(0) + e^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{11}$$ 2

Some tried this backwards.

(e) $x(t)$ can be found either by evaluating the convolution integral in (11), or by using Laplace transforms. The Laplace transform method requires a bit less effort (in this case):

i. *By convolution integral:* Since A is triangular, its eigenvalues can be seen by inspection to be $\lambda_1 = -1$ and $\lambda_2 = -2$ (the diagonal elements). Hence find the eigenvectors:

$$w_1 : \begin{bmatrix} -1 & 0 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} w_{11} \\ w_{21} \end{bmatrix} = -1\begin{bmatrix} w_{11} \\ w_{21} \end{bmatrix} \quad \text{and} \quad w_2 : \begin{bmatrix} -1 & 0 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} w_{12} \\ w_{22} \end{bmatrix} = -2\begin{bmatrix} w_{12} \\ w_{22} \end{bmatrix} \tag{12}$$

hence $2w_{11} = w_{21}$ and $w_{12} = 0$. So we have

$$w_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \quad w_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad W = \begin{bmatrix} w_1 & w_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & 0 \\ 2/\sqrt{5} & 1 \end{bmatrix} \tag{13}$$ 2

(the normalisation of w_1 is not really necessary) hence

$$W^{-1} = \begin{bmatrix} \sqrt{5} & 0 \\ -2 & 1 \end{bmatrix} \tag{14}$$

So we have, from (6):

$$e^{At} = \begin{bmatrix} 1/\sqrt{5} & 0 \\ 2/\sqrt{5} & 1 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} \sqrt{5} & 0 \\ -2 & 1 \end{bmatrix} \tag{15}$$ 2

Note that we don't need e^{At} on its own, since $x(0) = 0$, but we only need $e^{At}B$, which is quicker to evaluate:

$$\begin{aligned} e^{At}B &= \begin{bmatrix} 1/\sqrt{5} & 0 \\ 2/\sqrt{5} & 1 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} \sqrt{5} & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} & (16) \\ &= \begin{bmatrix} 1/\sqrt{5} & 0 \\ 2/\sqrt{5} & 1 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} \sqrt{5} \\ -2 \end{bmatrix} & (17) \\ &= \begin{bmatrix} 1/\sqrt{5} & 0 \\ 2/\sqrt{5} & 1 \end{bmatrix}\begin{bmatrix} \sqrt{5}e^{-t} \\ -2e^{-2t} \end{bmatrix} & (18) \\ &= \begin{bmatrix} e^{-t} \\ 2e^{-t} - 2e^{-2t} \end{bmatrix} & (19) \end{aligned}$$ 2

Now $u(\tau) = 1$ for $0 \le \tau \le t$, and $x(0) = 0$, so we get, using (11)

$$\begin{aligned} x(t) &= \int_0^t \begin{bmatrix} e^{\tau-t} \\ 2e^{\tau-t} - 2e^{2\tau-2t} \end{bmatrix} d\tau & (20) \\ &= \begin{bmatrix} e^{\tau-t} \\ 2e^{\tau-t} - e^{2\tau-2t} \end{bmatrix}_{\tau=0}^{\tau=t} & (21) \\ &= \begin{bmatrix} 1 - e^{-t} \\ 1 - 2e^{-t} + e^{-2t} \end{bmatrix} & (22) \end{aligned}$$ 2

Check: This gives $x(0) = 0$.

ii. *By Laplace transforms:* From $\dot{x} = Ax + Bu$ we get, by taking Laplace transforms:

$$(sI - A)X(s) = BU(s) + x(0) \tag{23}$$ 2

Since $x(0) = 0$ in this case, we get $X(s) = (sI - A)^{-1}BU(s)$. We have

$$\begin{aligned}(sI - A)^{-1} &= \begin{bmatrix} s+1 & 0 \\ -2 & s+2 \end{bmatrix}^{-1} && (24)\\ &= \frac{1}{(s+1)(s+2)}\begin{bmatrix} s+2 & 0 \\ 2 & s+1 \end{bmatrix} && (25)\end{aligned}$$ 2

hence

$$\begin{aligned}X(s) &= \frac{1}{(s+1)(s+2)}\begin{bmatrix} s+2 \\ 2 \end{bmatrix}\frac{1}{s} && (26)\\ &= \begin{bmatrix} \frac{1}{s(s+1)} \\ \frac{2}{s(s+1)(s+2)} \end{bmatrix} && (27)\\ &= \begin{bmatrix} \frac{1}{s} - \frac{1}{s+1} \\ \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2} \end{bmatrix} && (28)\end{aligned}$$ 2

Taking inverse Laplace transforms gives

$$x(t) = \begin{bmatrix} 1 - e^{-t} \\ 1 - 2e^{-t} + e^{-2t} \end{bmatrix} \quad \text{for} \quad t > 0. \tag{29}$$ 2

Check: This gives $x(0) = 0$.

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2. (a) Take the state vector to be $x = [\theta, \dot{\theta}, v]^T$. Then the state-space equation in standard form is

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} \theta \\ \dot{\theta} \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} u \tag{30}$$ 3

(b) Form the controllability matrix (with $n = 3$):

$$P = [B, AB, A^2B] = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{bmatrix} \tag{31}$$ 3

Evaluate the determinant: $\det(P) = -1(0+2) - 1(1-1) = -2 \neq 0$, so P has full rank, hence the system is controllable.

(c) The given feedback can be written as $u = -Kx$, where $K = [k_1, k_2, k_3]$. Hence the closed-loop system behaviour is given by $\dot{x} = Ax - BKx = (A - BK)x$, so the closed-loop poles are given by the eigenvalues of $A - BK$. We have 2

$$\begin{aligned}A - BK &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} && (32)\\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ k_1 & k_2 & k_3 \\ -k_1 & -k_2 & -k_3 \end{bmatrix} && (33)\\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1-k_1 & -k_2 & 1-k_3 \\ 1+k_1 & k_2 & -1+k_3 \end{bmatrix} && (34)\end{aligned}$$ 1

Substituting the given gain values: $K = [12, 9, 4]$ gives

$$A - BK = \begin{bmatrix} 0 & 1 & 0 \\ -11 & -9 & -3 \\ 13 & 9 & 3 \end{bmatrix} \tag{35}$$ 1

To find the eigenvalues, find the characteristic polynomial:

$$\begin{aligned}
\det(\lambda I - A + BK) &= \det \begin{bmatrix} \lambda & -1 & 0 \\ 11 & \lambda+9 & 3 \\ -13 & -9 & \lambda-3 \end{bmatrix} && (36)\\
&= \lambda[(\lambda+9)(\lambda-3)+27] + 1[11(\lambda-3)+39] && (37)\\
&= \lambda[\lambda^2+6\lambda] + [11\lambda+6] && (38)\\
&= \lambda^3+6\lambda^2+11\lambda+6 && (39)
\end{aligned}$$

2

If the closed-loop poles were at $-1, -2, -3$ then the closed-loop characteristic polynomial would be

$$(\lambda+1)(\lambda+2)(\lambda+3) = (\lambda+1)(\lambda^2+5\lambda+6) = \lambda^3+6\lambda^2+11\lambda+6 \quad (40)$$

which is the same as (39), so this verifies that the closed-loop poles are at the locations stated in the question. 2

Since these locations are all real and negative, this also verifies that the closed loop is asymptotically stable. (Alternatively, the Routh-Hurwitz test could be applied to (39), but this is not necessary if the pole locations are known.) OR PUT $\lambda = -1, -2, -3$ & check char poly $= 0$.

(d) *This can be answered by presenting the general scheme, then specialising to the case of the transporter, or by presenting the whole discussion specialised to the transporter.*

First variant: If a state-feedback scheme $u = -Kx$ is in place, integral action can be added as follows. Let $y = Cx$ define the outputs which are to be controlled without steady-state error, and let r be the vector of desired values of these outputs. Define additional integrator states x_I, such that $\dot{x}_I = e = r - y = r - Cx$. Now augment the state-feedback scheme so that $u = -Kx - K_I x_I$. 2
The ~~gains in K, as well as the~~ new ones in K_I, must be recomputed to get satisfactory closed-loop poles. The standard machinery of state-feedback design can be used to do this, starting with the augmented dynamics of the plant:

$$\begin{bmatrix} \dot{x} \\ \dot{x}_I \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} u \\ r \end{bmatrix} \quad (41)$$

2

If the closed-loop system is asymptotically stable then it will come to rest at an equilibrium with $\dot{x}_I = 0$, namely with $y = r$.

For the transporter, only the speed is to be controlled without steady-state error, so $C = [0, 0, 1]$. x_I consists of a single state variable, and only one integrator is required. Figure 1 shows the scheme in this case. 2

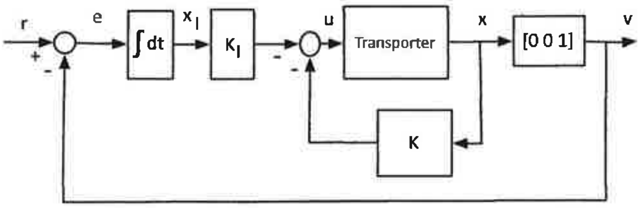

Figure 1: State feedback with integral action for the transporter

Second variant: The speed is to be controlled without steady-state error, so integral action should be applied to this variable only. Define an additional integrator state x_I, such that $\dot{x}_I = e = r - v$, 2
where r is the required speed. Note that this can be represented as $\dot{x}_I = r - Cx$, where $C = [0, 0, 1]$.
The previous state-feedback scheme is now augmented in the following way: $u = -Kx - K_I x_I$. The gains in K, as well as the new gain K_I, must be recomputed to get satisfactory closed-loop poles. 2
The standard machinery of state-feedback design can be used to do this, starting with the augmented

dynamics of the plant:

$$\begin{bmatrix} \dot{x} \\ \dot{x}_I \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} u \\ r \end{bmatrix} \tag{42}$$

2

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ r \end{bmatrix} \tag{43}$$

If the closed-loop system is asymptotically stable then it will come to rest at an equilibrium with $\dot{x}_I = 0$, namely with $y = r$. Figure 1 shows the resulting scheme.

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3. (a) Consider the damping of the two poles at the roots of $s^2 + s + 5$:

$$s^2 + s + 5 \equiv s^2 + 2\zeta\omega_n s + \omega_n^2 \tag{44}$$

gives $\omega_n = \sqrt{5}$ and $\zeta = 1/2\sqrt{5} = 0.224$. This damping is smaller than $1/\sqrt{2}$. The root-locus diagram shows that the two branches of the root locus starting at these poles move into regions of lower damping as the gain k increases. Explanation: They move towards a vertical asymptote (because there are 4 poles and 2 zeros) which is located at the 'centre of gravity' of the poles and zeros: 2 2

$$\frac{(0-1) - (-0.1 - 0.2)}{4-2} = -0.35 \tag{45}$$

So the closed-loop damping requirement will be violated with any value of k.

(b) i. Since $q = \dot{\theta}$, we have

$$H(s) = sG(s) = \frac{s(s+0.1)(s+0.2)}{(s^2+0.02)(s^2+s+5)} \tag{46}$$

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ii. There is now an additional zero at 0, with the other poles and zeros remaining unchanged. The portions of the real axis that are on the root-locus are now ('left of an odd number of poles and zeros'): between 0 and -0.1, and to the left of -0.2. The excess of poles over zeros is now $4 - 3 = 1$, so there is only one asymptote, and it lies along the real axis, towards negative values (angle π with the positive real axis). Each branch of the locus starts at a pole and ends on a zero or along the asymptote. *(Not really expected from candidates: By considering the 'angle condition' that is satisfied at each point on the root-locus, it is clear that each branch leaves each pole 'to the left'.)* The root-locus diagram must therefore look something like Fig.2. 2 2

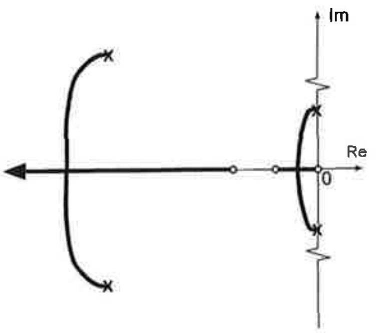

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Figure 2: Root-locus diagram for $H(s)$

iii. As k is increased, the damping of both pairs of complex poles increases, according to Fig.2, until all the closed-loop poles become real (over-damped). (Even if it does not increase monotonically, it must increase eventually, as the complex branches reach the real axis for large enough k.) There is therefore some value of k, beyond which the closed-loop damping requirement will be met. 3

(c) Consider the case when the pitch angle θ is fed back. The steady-state gain from the pilot command to θ is

$$\frac{G(0)}{1+kG(0)} = \frac{1/5}{1+k/5} = \frac{1}{5+k} \tag{47}$$ 1

So in this case the aircraft will settle to a pitch angle which depends on the value of the pilot command. If k is large, then the closed-loop gain from the command to θ will be small. The final pitch angle may also depend on any constant disturbances acting on the aircraft, as there is no integral action in the feedback loop. The damping will necessarily be low (from part (a)), so the settling to a constant pitch angle will follow some underdamped oscillations. The steady-state gain from the pilot command to the elevator angle δ is 1

$$\frac{1}{1+kG(0)} = \frac{1}{1+k/5} = \frac{5}{5+k} \tag{48}$$

so the elevator angle will settle to a value which depends on the pilot command and on k. 1

Now consider the case when the pitch rate q is fed back. In this case the steady-state gain from the pilot command to q is

$$\frac{H(0)}{1+kH(0)} = \frac{0}{1+k0} = 0 \tag{49}$$

So in this case the aircraft will settle to a constant pitch angle. However, this angle will depend on the previous command history, not just on the final value of the command — it will be approximately the pitch angle that the aircraft had when the pilot command became constant. The response will be well-damped in this case (assuming a suitable value of k has been implemented). The steady-state gain from the pilot command to the elevator angle δ is now 1 1

$$\frac{1}{1+kH(0)} = \frac{1}{1+k0} = 1 \tag{50}$$

so the elevator angle will settle to a value which depends on the pilot command but not on k.

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4. (a) The system is *observable* if, given measurements of the output $y(t)$, (and the input $u(t)$ if it is non-zero) over any finite interval $0 \le t \le T$, it is possible to deduce the initial state $x(0)$. 2

(Several alternative but equivalent definitions are possible. For example: '... it is possible to deduce the state $x(t)$ for $0 \le t \le T$'.)

Observability can be tested by checking the rank of the observability matrix

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{51}$$ 2

where n is the state dimension. The system is observable if and only if $\text{rank}(Q) = n$.

(b) We have

$$\begin{aligned} y(0) &= Cx(0) + Du(0) && (52)\\ \dot{y}(0) &= C\dot{x}(0) + D\dot{u}(0) = CAx(0) + CBu(0) + D\dot{u}(0) && (53)\\ \ddot{y}(0) &= C\ddot{x}(0) + D\ddot{u}(0) = CA^2x(0) + CABu(0) + CB\dot{u}(0) + D\ddot{u}(0) && (54)\\ &\vdots && (55)\\ y^{(n-1)}(0) &= Cx^{(n-1)}(0) + Du^{(n-1)}(0) = CA^{n-1}x(0) + CA^{n-2}Bu(0) + \cdots + CBu^{(n-2)}(0) + Du^{(n-1)}(0) && (56) \end{aligned}$$ 2

which can be re-arranged into

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x(0) = \begin{bmatrix} y(0) - Du(0) \\ \dot{y}(0) - CBu(0) - D\dot{u}(0) \\ \vdots \\ y^{(n-1)}(0) - CA^{n-2}Bu(0) - \cdots - CBu^{(n-2)}(0) - Du^{(n-1)}(0) \end{bmatrix} \tag{57}$$ 2

which is of the form

$$Qx(0) = v \tag{58}$$

where v is a known vector. (The assumption here is that u and y can be measured without noise, and hence their derivatives can be known perfectly.) But (58) has a unique solution if Q has full column rank. Q has n columns, so Q has full column rank if $\text{rank}(Q) = n$. Hence $x(0)$ can be determined uniquely if $\text{rank}(Q) = n$. Hence the system is observable if $\text{rank}(Q) = n$. 2 2

(c) We have

$$CA = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -100 & 100 & 0 & 0 \\ 100 & 0 & 0 & 0 \end{bmatrix} \tag{59}$$

and $n = 4$ so that the observability matrix is

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -100 & 100 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{60}$$

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and it is not necessary to look at further rows, because it is clear that these first 4 rows are linearly independent. Hence $\text{rank}(Q) = 4$, and the system is observable.

(d) If the altitude rate measurement, namely the third output, is not available, then the matrix C must be replaced by the matrix consisting of only the first two rows of C:

$$C_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{61}$$

2

In this case the observability matrix is obtained by removing rows 3, 6, 9, 12 from the previously calculated Q, or explicitly by computing

$$Q_2 = \begin{bmatrix} C_2 \\ C_2A \\ C_2A^2 \\ C_2A^3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -100 & 100 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{62}$$

2

Again it is not necessary to compute further rows, as it is clear that these first 4 are linearly independent, so that $\text{rank}(Q_2) = 4$, and hence the system remains observable.

3F2 Systems and Control: 2012 Answers

Prof. J.M. Maciejowski

7 June 2012

1. (a) —
   (b) —
   (c) —
   (d) —
   (e)
   $$x(t) = \begin{bmatrix} 1 - e^{-t} \\ 1 - 2e^{-t} + e^{-2t} \end{bmatrix}$$

2. (a)
   $$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} u$$
   (b) —
   (c) —
   (d) —

3. (a) —
   (b) i.
   $$H(s) = \frac{s(s+0.1)(s+0.2)}{(s^2+0.02)(s^2+s+5)}$$
   ii. —
   iii. —
   (c) —

4. (a) —
   (b) —
   (c) —
   (d) Observable.