1.

Data Transmission

Engineering Tripos Part II A. Module 3F4. 2011/12.

1. a) $$S_x(\omega) = \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{T}$$ (1)

The truncated signal $x_T(t)$ is

$$x_T(t) = \sum_{n=-N}^{N} a_n \, \delta(t - nT_s), \qquad T = (2N+1)T_s$$

Now,

$$X_T(\omega) = \int_{-\infty}^{\infty} x_T(t)\, e^{-j\omega t}\, dt$$

and substituting for $x_T(t)$ gives,

$$X_T(\omega) = \int_{-\infty}^{\infty} \sum_{n=-N}^{N} a_n \, \delta(t - nT_s)\, e^{-j\omega t}\, dt$$

$$= \sum_{n=-N}^{N} a_n \int_{-\infty}^{\infty} \delta(t - nT_s)\, e^{-j\omega t}\, dt$$

$$= \sum_{n=-N}^{N} a_n \, e^{-jn\omega T_s}$$

Now,

$$|X_T(\omega)|^2 = X_T(\omega)\, X_T^*(\omega)$$

$$= \sum_{n=-N}^{N} a_n e^{-jn\omega T_s} \sum_{k=-N}^{N} a_k e^{jk\omega T_s}$$

$$= \sum_{n=-N}^{N} \sum_{k=-N}^{N} a_n a_k \, e^{j(k-n)\omega T_s}$$

Now,

$$E[|X_T(\omega)|^2] = E\left[\sum_{n=-N}^{N} \sum_{k=-N}^{N} a_n a_k \, e^{j(k-n)\omega T_s}\right]$$

$$= \sum_{n=-N}^{N} \sum_{k=-N}^{N} E[a_n a_k]\, e^{j(k-n)\omega T_s}$$

Let $k = n + m$ and so $m = k - n$,

$$E[|X_T(\omega)|^2] = \sum_{n=-N}^{N} \sum_{m=-N-n}^{N-n} R(m)\, e^{jm\omega T_s}$$

where, $R(m) = E[a_n a_{n+m}]$

Replace the outer sum over index n
by $(2N+1)$ gives

$$E\left[\,|X_T(\omega)|^2\,\right] = (2N+1)\sum_{m=-N-n}^{N-n} R(m)\, e^{jm\omega T_s}$$

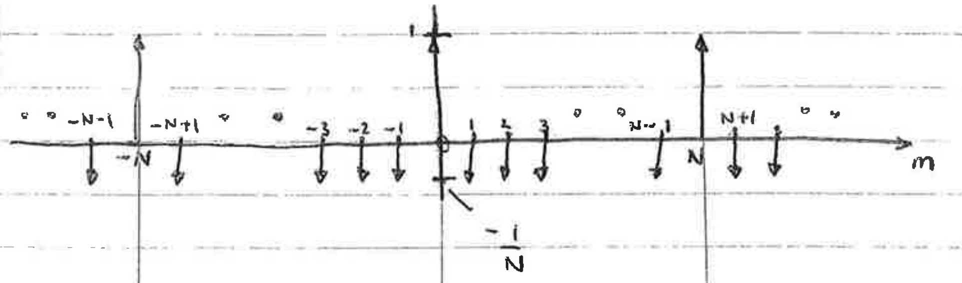
So from ①

$$S_x(\omega) = \lim_{N\to\infty} \frac{1}{(2N+1)T_s}(2N+1)\sum_{m=-N-n}^{N-n} R(m)\, e^{jm\omega T_s}$$

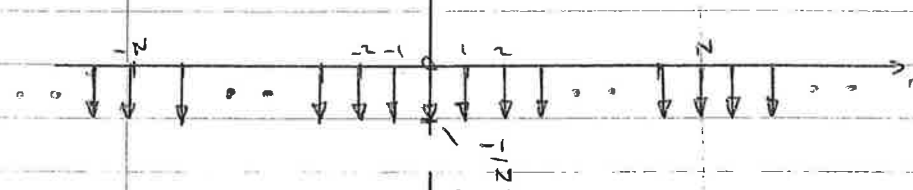
$$S_x(\omega) = \frac{1}{T_s}\sum_{m=-\infty}^{\infty} R(m)\, e^{jm\omega T_s}$$

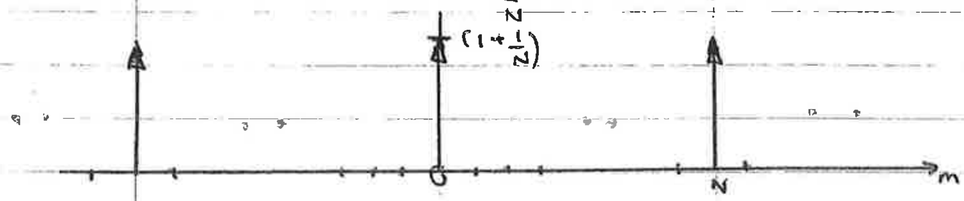
where $R(m) = E[\,a_n\, a_{n+m}\,]$ [8]

b) The discrete ACF has the following form,



That can be expressed as the sum of the following constant value ACFs:





So,

$$S_x(\omega) = \frac{1}{T_s}\left[\sum_{m=-\infty}^{\infty}\left(1+\frac{1}{N}\right)e^{jNmT_s\omega} + \sum_{k=-\infty}^{\infty}\left(\frac{-1}{N}\right)e^{jkT_s\omega}\right]$$

$$S_x(\omega) = \frac{1}{T_s}\left[\frac{1+N}{N}\sum_{m=-\infty}^{\infty} e^{jmNT_s\omega} - \frac{1}{N}\sum_{k=-\infty}^{\infty} e^{jkT_s\omega}\right],$$

Now use following identity:

$$\sum_{m=-\infty}^{\infty} e^{jmT_s\omega} = \frac{2\pi}{T_s}\sum_{m=-\infty}^{\infty}\delta\left(\omega - m\frac{2\pi}{T_s}\right)$$

So,

$$S_x(\omega) = \frac{1}{T_s}\left[\frac{1+N}{N}\cdot\frac{2\pi}{NT_s}\sum_{m=-\infty}^{\infty}\delta\left(\omega - \frac{m2\pi}{NT_s}\right) - \frac{1}{N}\cdot\frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta\left(\omega - k\frac{2\pi}{T_s}\right)\right]$$

$$S_x(\omega) = \frac{2\pi(1+N)}{N^2 T_s^2} \sum_{m=-\infty}^{\infty} \delta\left(\omega - \frac{m\,2\pi}{N T_s}\right) - \frac{2\pi}{N T_s^2} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T_s}\right)$$ 2

[8]

(c) To get the PSD at the filter output,

$$S_y(\omega) = S_x(\omega)\,|H(\omega)|^2$$ 1

From data book,

$$H(\omega) = T_s\,\mathrm{sinc}\left(\frac{\omega T_s}{2}\right)$$ 1

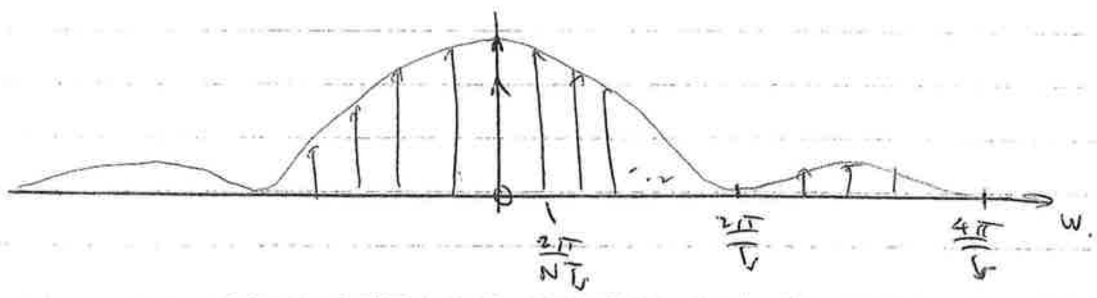
which has zero crossings at $\frac{2\pi}{T_s}$.
Consequently only the d.c. ~~term~~ component of the 2nd term of $S_x(\omega)$ will contribute, so,

$$S_y(\omega) = \frac{2\pi(1+N)}{N^2 T_s^2} \sum_{m=-\infty}^{\infty} \delta\left(\omega - \frac{m\,2\pi}{N T_s}\right) |H(\omega)|^2 - \frac{2\pi}{N T_s^2}\,\delta(\omega)\,|H(\omega)|^2$$

$$S_y(\omega) = \frac{2\pi(1+N)}{N^2 T_s^2} \sum_{m=-\infty}^{\infty} \delta\left(\omega - \frac{m\,2\pi}{N T_s}\right)\left(T_s^2\,\mathrm{sinc}^2\left(\frac{\omega T_s}{2}\right)\right) - \frac{2\pi}{N T_s^2}\cdot T_s^2\,\delta(\omega)$$

$$S_y(\omega) = \frac{2\pi(1+N)}{N^2 T_s^2}\cdot T_s^2 \sum_{m=-\infty}^{\infty} \delta\left(\omega - \frac{m\,2\pi}{N T_s}\right) \mathrm{sinc}^2\left(\frac{m\,2\pi}{N T_s}\cdot\frac{T_s}{2}\right) - \frac{2\pi}{N}\,\delta(\omega)$$

$$S_y(\omega) = \frac{2\pi(1+N)}{N^2} \sum_{m=-\infty}^{\infty} \mathrm{sinc}^2\left(\frac{m\pi}{N}\right) \delta\left(\omega - \frac{m\,2\pi}{N T_s}\right) - \frac{2\pi}{N}\,\delta(\omega)$$ 1



1 [4]

2. a) The structure of a code word is as follows,

codebit 5 4 3 2 1 0

$$[d_2 \; d_1 \; d_0 \; p_2 \; p_1 \; p_0]$$

The code words are generated using, where $n = 6$ and $k = 3$

$$\underset{(1\times n)}{c} = \underset{(1\times k)}{d}\;\underset{(k\times n)}{G}$$

and,

$$G = [\underset{(k\times k)}{I} \;\vdots\; \underset{(k\times(n-k))}{P}]$$

$$\underset{c}{[c_5\, c_4\, c_3\, c_2\, c_1\, c_0]} = \underset{d}{[d_2\; d_1\; d_0]} \underset{G}{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]}$$ 3

We also know,

$$\underset{(n-k)\times n}{H} = [\underset{(n-k)\times k}{-P^T} \;\vdots\; \underset{(n-k)\times(n-k)}{I}]$$

so,

$$H = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$ 3

Alternatively we can get H directly using syndrome calculator,

$$\underset{(1\times n-k)}{s} = \underset{(1\times n)}{c}\;\underset{(n\times(n-k))}{H^T}$$

$$s = [c_5\ c_4\ c_3\ c_2\ c_1\ c_0] \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ 5, 4 \text{ and } 2 & 4, 3, 1 & 5, 3, 0 \end{array}$$

$$\text{so}\quad H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

[6]

which is the same as before.

b)

Data word	codeword	distance
0 0 0	0 0 0 0 0 0	-
0 0 1	0 0 1 0 1 1	3
0 1 0	0 1 0 1 1 0	3
0 1 1	0 1 1 1 0 1	4
1 0 0	1 0 0 1 0 1	3
1 0 1	1 0 1 1 1 0	4
1 1 0	1 1 0 0 1 1	4
1 1 1	1 1 1 0 0 0	3

So $d_{min} = 3$

Max number of correctable errors $= \left\lfloor \dfrac{d_{min} - 1}{2} \right\rfloor$

$$= \left\lfloor \frac{3-1}{2} \right\rfloor = \underline{\underline{1}}$$

(6)

c) So what are the syndromes corresponding to all single bit error patterns?

$$\underset{(1\times3)}{s} = \underset{(1\times6)}{e}\,\underset{(6\times3)}{H^T}$$

$$s = e \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

e	s
0 0 0 0 0 0	0 0 0
0 0 0 0 0 1	0 0 1
0 0 0 0 1 0	0 1 0
0 0 0 1 0 0	1 0 0
0 0 1 0 0 0	0 1 1
0 1 0 0 0 0	1 1 0
1 0 0 0 0 0	1 0 1

See we actually have $2^3 = 8$ possible syndromes, i.e., enough for no errors and 6 single bit errors with 1 left over.

We can assign this to identify a number of 2-bit errors. Syndrome not used so far is 1 1 1.

One of the detectable 2-bit patterns that corresponds with syndromes

0 1 1   0 0 1 0 0 0

and 1 0 0   0 0 0 1 0 0

ie, error pattern

0 0 1 1 0 0 .

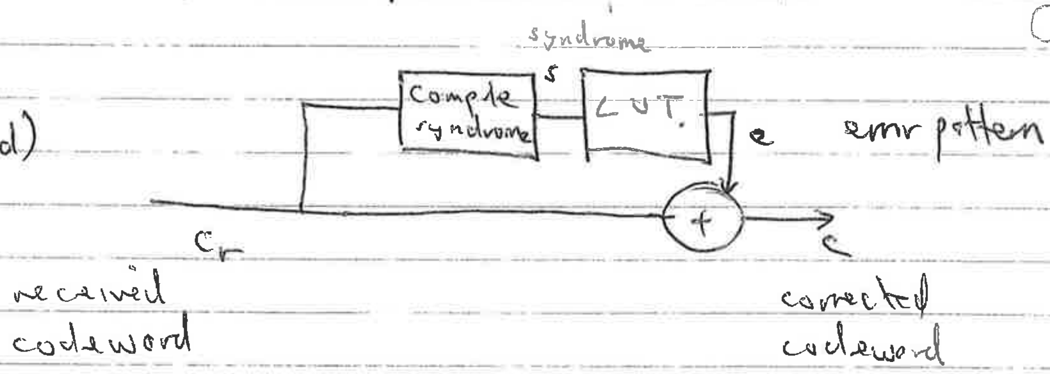
There are lots of others! – so we cannot use it for correction, ie, only detection of some 2-bit error patterns.

Standard Array

		c2	...	c_8	Syndrome s
c_1	0 0 0 0 0 0	c2	...	c_8	0 0 0
e1	0 0 0 0 0 1	c_2+e_1	...	c_8+e_1	0 0 1
e2	0 0 0 0 1 0	c_2+e_2	...	c_8+e_2	0 1 0
e3	0 0 0 1 0 0	⋮	...	⋮	1 0 0
e4	0 0 1 0 0 0	⋮	...	⋮	0 1 1
e5	0 1 0 0 0 0	⋮	...	⋮	1 1 0
e6	1 0 0 0 0 0	c_2+e_6	...	c_8+e_6	1 0 1
e7	0 0 1 1 0 0	c_2+e_7	...	c_8+e_7	1 1 1

2

[5]

d) 

[3]

3. (a) In real terms

$$s(t) = a(t)\cos(\omega_c t + \phi(t)) \quad 1$$

where $a(t)$ introduces amplitude modulation & $\phi(t)$ produces phase or frequency modulation.

To combine these two effects into a phasor waveform $p(t)$:

$$\text{Let} \quad s(t) = Re\left[a(t)\, e^{j(\omega_c t + \phi(t))}\right]$$

$$= Re\left[a(t)\, e^{j\phi(t)} \cdot e^{j\omega_c t}\right]$$

$$= Re\left[p(t)\, e^{j\omega_c t}\right] \quad 1$$

where $p(t) = a(t)\, e^{j\phi(t)}$, 1 & hence represents both phase, frequency & amplitude modulation.

Note that for an input signal $x(t)$,

$\phi(t) \propto x(t)$ gives phase modulation

and $\phi(t) \propto \frac{dx}{dt}$ gives frequency modulation.

and $a(t) \propto x(t)$ gives amplitude modulation.

1

[4]

3. (b) Rewriting $s(t)$ as:

$$s(t) = \frac{1}{2}\left(p(t)\,e^{j\omega_c t} + p^*(t)\,e^{-j\omega_c t} \right)$$

where p^* is the complex conjugate of p, we can now take Fourier-transforms ~~to get~~ as follows:

$$p(t) \Longleftrightarrow P(\omega)$$

$$p(t)\,e^{j\omega_c t} \Longleftrightarrow P(\omega - \omega_c)$$

$$p^*(t) \Longleftrightarrow P^*(-\omega)$$

$$p^*(t)\,e^{-j\omega_c t} \Longleftrightarrow P^*(-(\omega + \omega_c))$$

$$\therefore\; S(\omega) = \frac{1}{2}\left[P(\omega-\omega_c) + P^*(-(\omega+\omega_c)) \right] \qquad [4]$$

(c) For both PSK & QAM, the output ~~symbols~~ phase waveform $p(t)$ can be regarded as the convolution of a stream of data impulses $b(t)$, with a pulse-shaping filter $g(t)$. Hence in the frequency domain

$$P(\omega) = B(\omega)\,.\,G(\omega)$$

& $$|P(\omega)|^2 = |B(\omega)|^2 . |G(\omega)|^2$$

For rectangular symbols of duration $T_s = mT_b$, $g(t)$ is a rectangular pulse of duration T_s.

3 (c) (cont.)

Hence $|G(\omega)|^2 \propto \operatorname{sinc}^2\left(\frac{\omega T_s}{2}\right)$

If the data impulses in $b(t)$ are random, then the discrete ACF of both PSK and QAM impulses will just be a single δ-function at ~~zero~~ $\tau = 0$, so the power spectrum $|B(\omega)|^2$ will be flat (= a constant at all frequencies).

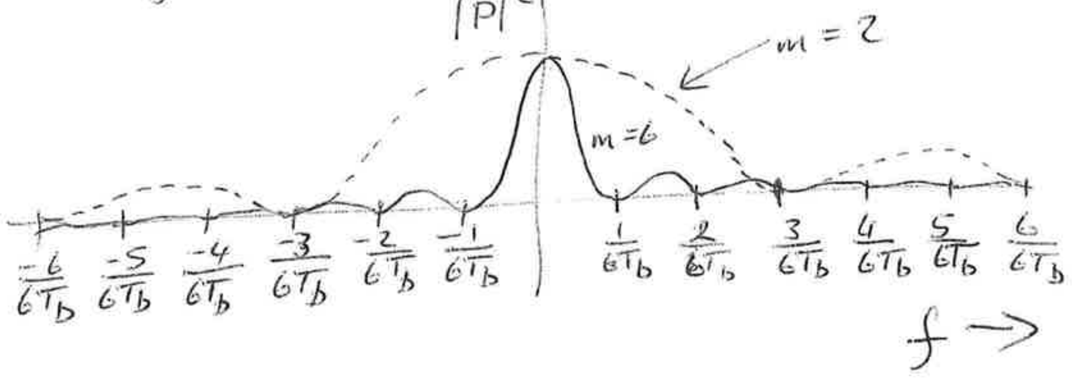
$$\therefore \quad |P(\omega)|^2 \propto |G(\omega)|^2 \propto \operatorname{sinc}^2\left(\frac{\omega m T_b}{2}\right)$$

The main lobe of this sinc² function will extend from $\frac{\omega m T_b}{2} = -\pi$ to $+\pi$.

Hence $f m T_b$ goes from -1 to $+1$

and f goes from $-\frac{1}{mT_b}$ to $+\frac{1}{mT_b}$.

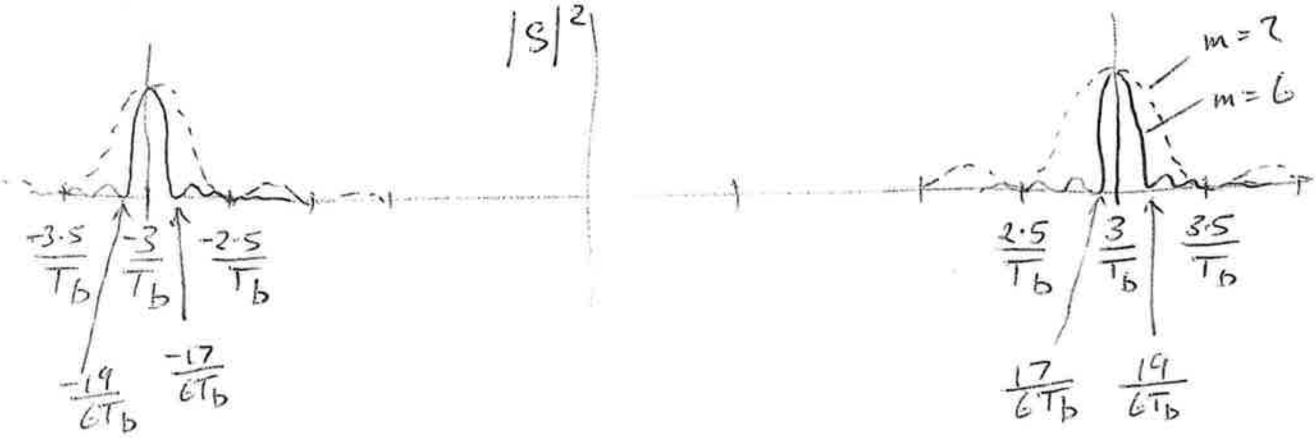
So for $m = 2$ & $m = 6$, the power spectra $|P(\omega)|^2$ will be as follows:



3(c) (cont.)

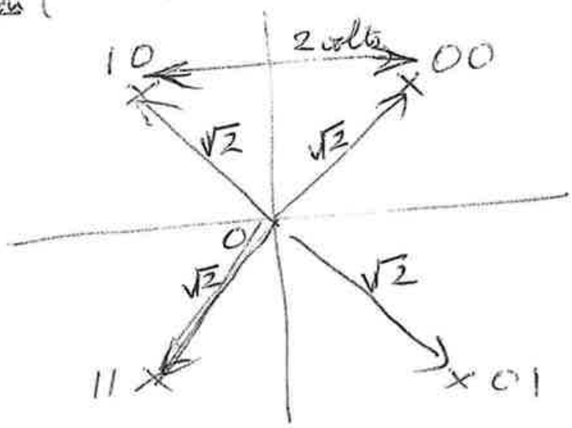
When modulated onto a carrier of 3 × the bit rate: $f_c = \frac{3}{T_b}$

& the spectrum of $|P(\omega)|^2$ gets shifted by $\pm \frac{3}{T_b}$ Hz above and below zero frequency by being modulated onto the carrier. Hence the spectra become:



[6]

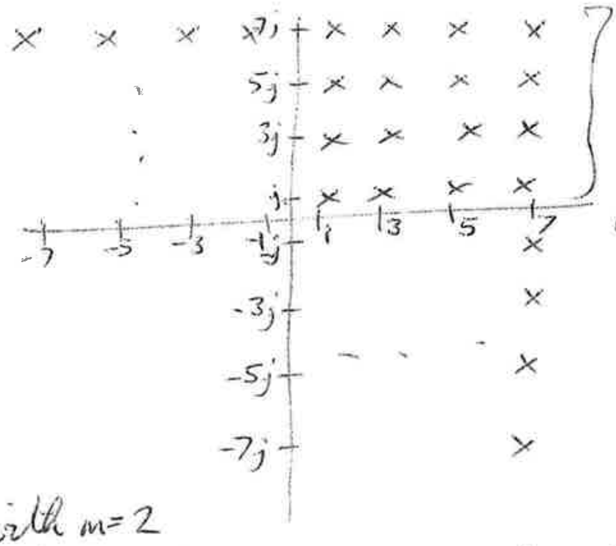
3(d) <u>QPSK constellation</u> $(m = 2)$



1

(or it can be rotated 45° from this position)

QAM constellation $(m = 6)$



repeated in all 4 quadrants with 64 states.

1

For QPSK with $m=2$ if the mean squared signal voltage per symbol is 2 volt then the distance of each of the 4 points from 0 is $\sqrt{2}$ volt. Hence the states are 2 volts apart & the minimum noise vector amplitude that will just take a state across a decision threshold is <u>1 volt</u>.

1

3(d) (cont)

For 64-QAM, ~~the~~ assume the constellation points are at $-7s, -5s, \ldots, 5s, 7s$ in each direction (real & imag.). The mean squared amplitude of the real component is then, when $M = 8$:

$$\frac{1}{8} \sum_{i=0}^{7} (2i+1-M)^2 . s^2 = \left(\frac{M^2 - 1}{3} \right) s^2 = \frac{63}{3} s^2 = 21 s^2$$

Hence the mean-squared amplitude of both components together is just $2 . 21 s^2 = 42 s^2$

Hence for 6 volts^2 of m.s. voltage: $42 s^2 = 6$ (since $m = 6$)

$\therefore s = \frac{1}{\sqrt{7}}$

The ~~max~~ minimum noise voltage to cause a decision error is then s (pts are spaced $2s$ apart),

so min. noise vector ampl. $= \frac{1}{\sqrt{7}} = 0.378$ volt. 1

Thus we see that, although the QAM system only requires $\frac{1}{3}$ of the bandwidth of the QPSK system, for the same bit rate $\frac{1}{T_b}$, QAM is much more sensitive to noise, requiring only 0.378 volts to cause an error instead of 1 volt for QPSK. 1

For digital video with fixed directional antenae, & high bit rates, QAM is more suitable in order to save bandwidth, but for digital audio to mobile antennae, it is better to use QPSK, to get max. resilience to noise. (c) 1

4. (a) Analogue Modulation

1. Simple hardware
2. Relatively efficient use of bandwidth 1

BUT

3. When noise is introduced on a link, it is very difficult to remove it afterwards.
4. Dynamic range limited by added noise.

Digital Modulation

1. Less susceptible to cumulative degradations on several links in series, especially if error-correcting codes are used 1
2. Noise levels are determined by accuracy of A-D converters, not by the channel. 1
3. Many multiplexing / modulation options are available.
4. Bandwidth of digital systems tends to be large if compression is not used, but can be ~~mad~~ smaller than analogue if ~~with~~ good compression algorithms are used. 1
5. Many types of source material can be digitally encoded.
6. Complexity of digital systems is high, but still quite cheap with VLSI chips. The capability of modern DSP systems now makes digital systems more efficient than analogue & hence more cost effective for providers. 1 (5)

10.

4 (b) Multiple transmit paths is known as "multipath" and ~~the~~ the range of path delays produced, can cause different 1
transmitted symbols to be received at the same time, especially if the symbol rate is high. This is known as inter-symbol interference (ISI) & can make a received signal unusable. 1

OFDM distributes a high bit-rate signal across 1000s of carriers, so that the symbol rate on each carrier is low-enough for ISI not to be a problem. 1 Guard bands and Fourier analysis methods are used to allow all symbols to be decoded without ISI, because they are orthogonal to each other (hence orthogonal frequency division multiplexing). 1

If the guard band period can be made ≥ max delay difference between signals from ~~nearby~~ adjacent transmitters, then all transmitters in the UK can use the same frequency for a given channel, which avoids expensive frequency re-use plans being needed — saving typically 4:1 in bandwidth.

1

(5)

4. (c) For differential path delays up to 300μs, the guard period must be $\geq$ 300μs.

Hence total period for transmission of each symbol must be $\geq 4 \times 300 = 1200$ μs.

The min. analysis period for the receiver FFT is therefore $1200 - 300 = 900$ μs.

Hence the max freq spacing of carriers is $\frac{1}{900\mu} = 1111$ Hz. 1

For a 2 Mb/s audio data stream, and rate 1:2 error correction, the transmitted bit rate must be 4 Mb/s. 1

With QPSK modulation, 2 bits are transmitted per symbol, so each carrier can transmit 2 bits in 1.2 ms, which is 1667 bits/s. 1

$$\therefore \text{No of carriers} = \frac{4.10^6}{1667} = 2400$$ 1

$$\therefore \text{RF bandwidth} = 2400 \times \frac{1}{900\mu}$$

$$= \underline{2.67 \text{ MHz}}$$ 1

(5)

4(d) No of ~~audio~~ channels at 128 Kbit/s in a 2 Mb/s data stream $= \frac{2.10^6}{128.10^3}$

$= 15$ (rounding <u>down</u> to nearest integer)

Hence bandwidth per channel $= \frac{2.67\,\text{MHz}}{15}$

$= \underline{\underline{178\ \text{kHz}}}$.

An analogue signal with peak deviation of 100 kHz on <u>each side</u> of the carrier needs at least <u>200kHz</u> of R.F. bandwidth, ignoring the signal frequency dependent term in Carson's rule, which is probably small if the max deviation occurs at low ~~signal~~ audio frequencies. Hence the digital system needs about 0.9 of the analogue bandwidth per channel.

<u>BUT</u> with single-frequency coverage of OFDM replacing the typical 4:1 frequency allocation plan of analogue, the digital system needs only $0.25 \times 0.9 = 0.23$ of the analogue bandwidth per channel.

(5)

IJ Wassell / N.G. Kingsbury 8-5-12.

Engineering Triops Part 2A
Module 3F4. Data Transmission, May 2012 - Answers

1.
a)
b)

$$S_x(\omega) = \frac{2\pi(1+N)}{N^2T_s^2}\sum_{m=-\infty}^{\infty}\delta\left(\omega - \frac{m2\pi}{NT_s}\right) - \frac{2\pi}{NT_s^2}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{k2\pi}{T_s}\right)$$

c)

$$S_y(\omega) = \frac{2\pi(1+N)}{N^2}\sum_{m=-\infty}^{\infty}\operatorname{sinc}^2\left(\frac{m\pi}{N}\right)\delta\left(\omega - \frac{m2\pi}{NT_s}\right) - \frac{2\pi}{N}\delta(\omega)$$

2.
a)

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

b) $d_{min} = 3$ Maximum number of correctable errors = 1

c)

c_1	000000	c_2	...	c_8	000
e_1	000001	c_2+e_1	...	c_8+e_1	001
e_2	000010	c_2+e_2	...	c_8+e_2	010
e_3	000100	c_2+e_3	...	c_8+e_3	100
e_4	001000	c_2+e_4	...	c_8+e_4	011
e_5	010000	c_2+e_5	...	c_8+e_5	110
e_6	100000	c_2+e_6	...	c_8+e_6	101
e_7	001100				111

Note: various 2 bit error patterns will map to syndrome 111. So can detect some 2 bit errors but cannot correct them.

3.
a)
b)
c)
d) QPSK = 1V, 64-QAM = 0.378V

4.
a)
b)
c) 1111Hz. 2.67MHz
d)