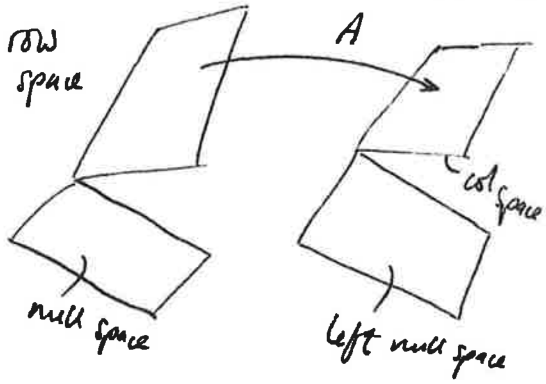
1

(a) The dimension of row space $= r =$ the rank of $(A) =$ the dimension of column space. The mapping A: row space $\rightarrow$ col space is, therefore, invertible. A^+ is a matrix which has this property i.e. $A^+A\underline{x} = \underline{x}$ for every $\underline{x} \in$ row space.



[15%]

(b) $$A^tA = \begin{bmatrix} 1 & 2 & -3 \\ -3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & 2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & -2 \\ -2 & 14 \end{bmatrix}$$

e-values $(14-\lambda)^2 - 4 = 0 \Rightarrow \lambda = 14 \pm 2 = 16, 12 \Rightarrow \sigma_1 = 4 \quad \sigma_2 = 2\sqrt{3}$

e-vectors

$16: \quad 14x - 2y = 16x \Rightarrow \underline{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -1/\sqrt{2} \end{bmatrix}$

$12: \quad 14x - 2y = 12x \Rightarrow \underline{q}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$\hat{\underline{q}}_1 = \frac{1}{\sigma_1} A\underline{q}_1 = \frac{1}{4\sqrt{2}}\begin{bmatrix} 1 & -3 \\ 2 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{4\sqrt{2}}\begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\hat{\underline{q}}_2 = \frac{1}{\sigma_2} A\underline{q}_2 = \frac{1}{2\sqrt{3}} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -3 \\ 2 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2\sqrt{6}}\begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$$\hat{\underline{q}}_3 \perp \hat{\underline{q}}_1, \hat{\underline{q}}_2 \Rightarrow \hat{\underline{q}}_3 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & -1 \\ -1 & 2 & -1 \end{vmatrix} \frac{1}{\sqrt{2}\sqrt{6}} = \frac{1}{2\sqrt{3}}\begin{bmatrix} +2 \\ +2 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 2\sqrt{3} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T \text{ or } \sigma_1 \hat{\underline{q}}_1 \underline{q}_1^T + \sigma_2 \hat{\underline{q}}_2 \underline{q}_2^T$$

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$$\Rightarrow A^{+} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & \frac{1}{2\sqrt{3}} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}^{T} = \begin{bmatrix} \frac{1}{24} & \frac{1}{6} & -\frac{5}{24} \\ -\frac{5}{24} & \frac{1}{6} & \frac{1}{24} \end{bmatrix}$$ [50%]

(c) S symmetric ⇒ e-vectors orthogonal & can take as unit vectors

∴ $SQ = Q\Lambda$ where Q = orthogonal matrix with e-vectors as columns
Λ = diagonal matrix " e-values down diagonal.

Q orthogonal ⇒ $Q^{-1} = Q^{t}$

∴ $S = Q\Lambda Q^{t}$ [10%]

(d) Singular value decomposition on B gives

B square ⇒ Σ square

$$B = \hat{Q}\Sigma Q^{t} = \hat{Q}\underbrace{Q^{t}Q}_{I}\Sigma Q^{t}$$

$\hat{Q}, Q$ orthogonal

$$= \underbrace{\hat{Q}Q^{t}}_{Q_1}\underbrace{(Q\Sigma Q^{t})}_{T}$$

Q_1 = product of orthogonal matrices ⇒ orthogonal

T is symmetric (see part (c)) with cols of Q as e-vectors and e-values = principal values of B (These can not be -ve)

[25%]

Principal assessor's comments:
A popular question. The standard linear algebra operations (principally SVD) were generally well done, but candidates showed a less good understanding of their underlying meanings and properties. Very few candidates recognised the significance of the result in part (c) to the solution of part (d).

Question 2

(a) From Fig. 1: $$L^2 = h^2 + \left(\frac{s}{2}\right)^2$$

$$\therefore \quad L^2 = 1^2 + \left(\frac{1.5}{2}\right)^2 = 1.5625 \quad \Rightarrow \quad L = 1.25 \text{ m}$$

Also from Fig. 1: $$\cos\beta = \frac{s}{2L} = \frac{1.5}{2\times 1.25} = 0.6$$

$$\sin\beta = \frac{h}{L} = \frac{1}{1.25} = 0.8$$

$$P(x,y) = \frac{EA}{2L}\left[\left(x\cos\beta + y\sin\beta\right)^2 + \left(-x\cos\beta + y\sin\beta\right)^2\right] - W\left[x\cos\theta + y\sin\theta\right]$$

$$\therefore \quad P(x,y) = \frac{EA}{2L}\left[2x^2\cos^2\beta + 2y^2\sin^2\beta\right] - W\left[x\cos\theta + y\sin\theta\right]$$

$$\therefore \quad P(x,y) = \frac{200\times 10^9 \times 10^{-5}}{2\times 1.25}\left[2x^2\times 0.6^2 + 2y^2\times 0.8^2\right] - 10^4\left[x\cos 30 + y\sin 30\right]$$

$$\therefore \quad P(x,y) = 5.76\times 10^5 x^2 + 1.024\times 10^6 y^2 - 8.66\times 10^3 x - 5\times 10^3 y$$ [15%]

(b) The common scaling factor of 10^3 can be eliminated without changing the nature of the problem:

$$\therefore \quad P(x,y) = 576x^2 + 1024y^2 - 8.66x - 5y$$

$$\therefore \quad \frac{\partial P}{\partial x} = 1152x - 8.66 \qquad (1)$$

$$\therefore \quad \frac{\partial^2 P}{\partial x^2} = 1152 \qquad (2)$$

$$\therefore \quad \frac{\partial P}{\partial y} = 2048y - 5 \qquad (3)$$

$$\therefore \quad \frac{\partial^2 P}{\partial y^2} = 2048 \qquad (4)$$

$$\therefore \quad \frac{\partial^2 P}{\partial x \partial y} = 0 \qquad (5)$$

From (1): $$\frac{\partial P}{\partial x} = 1152x - 8.66 = 0 \quad \Rightarrow \quad x = \frac{8.66}{1152} = 7.52\times 10^{-3} \text{ m}$$

From (3): $$\frac{\partial P}{\partial y} = 2048y - 5 = 0 \quad \Rightarrow \quad y = \frac{5}{2048} = 2.44\times 10^{-3} \text{ m}$$

From (2), (4) and (5) the Hessian: $$\boldsymbol{H} = \begin{bmatrix} 1152 & 0 \\ 0 & 2048 \end{bmatrix}$$

By inspection $\boldsymbol{H}$ is positive definite, so this solution is a minimum. [20%]

(c) Let $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k$

$$f(\mathbf{x}_{k+1}) = f(\mathbf{x}_k + \alpha_k \mathbf{d}_k) = f(\mathbf{x}_k) + \alpha_k \nabla f(\mathbf{x}_k)^T \mathbf{d}_k + \frac{\alpha_k^{\ 2}}{2} \mathbf{d}_k^{\ T} \boldsymbol{H}(\mathbf{x}_k) \mathbf{d}_k + R$$

Neglecting R and differentiating with respect to α_k:

$$\frac{\partial f}{\partial \alpha_k}(\mathbf{x}_k + \alpha_k \mathbf{d}_k) = \nabla f(\mathbf{x}_k)^T \mathbf{d}_k + \alpha_k \mathbf{d}_k^{\ T} \boldsymbol{H}(\mathbf{x}_k) \mathbf{d}_k$$

For a minimum:

$$\frac{\partial f}{\partial \alpha_k}(\mathbf{x}_k + \alpha_k \mathbf{d}_k) = \nabla f(\mathbf{x}_k)^T \mathbf{d}_k + \alpha_k \mathbf{d}_k^{\ T} \boldsymbol{H}(\mathbf{x}_k) \mathbf{d}_k = 0$$

$$\therefore \quad \alpha_k = -\frac{\nabla f(\mathbf{x}_k)^T \mathbf{d}_k}{\mathbf{d}_k^{\ T} \boldsymbol{H}(\mathbf{x}_k) \mathbf{d}_k}$$ [15%]

(d) From (b): $\nabla f(x,y) = \begin{bmatrix} 1152x - 8.66 \\ 2048y - 5 \end{bmatrix}$

For the SDM: $\mathbf{d} = -\nabla f = \begin{bmatrix} 8.66 - 1152x \\ 5 - 2048y \end{bmatrix}$

$$\therefore \quad \alpha_k = \frac{\mathbf{d}_k^{\ T} \mathbf{d}_k}{\mathbf{d}_k^{\ T} \boldsymbol{H}(\mathbf{x}_k) \mathbf{d}_k}$$

From (b): $\boldsymbol{H} = \begin{bmatrix} 1152 & 0 \\ 0 & 2048 \end{bmatrix}$

If $\mathbf{x}_1 = (0,0)$: $\mathbf{d}_1 = \begin{bmatrix} 8.66 \\ 5 \end{bmatrix}$

$$\therefore \quad \alpha_1 = \frac{8.66^2 + 5^2}{\begin{bmatrix} 8.66 & 5 \end{bmatrix} \begin{bmatrix} 1152 & 0 \\ 0 & 2048 \end{bmatrix} \begin{bmatrix} 8.66 \\ 5 \end{bmatrix}}$$

$$\therefore \quad \alpha_1 = \frac{8.66^2 + 5^2}{1152 \times 8.66^2 + 2048 \times 5^2} = 7.267 \times 10^{-4}$$

$$\therefore \quad \mathbf{x}_2 = \mathbf{x}_1 + \alpha_1 \mathbf{d}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 7.267 \times 10^{-4} \begin{bmatrix} 8.66 \\ 5 \end{bmatrix} = \begin{bmatrix} 6.294 \times 10^{-3} \\ 3.634 \times 10^{-3} \end{bmatrix}$$

$$\therefore \quad \mathbf{d}_2 = \begin{bmatrix} 8.66 - 1152 \times 6.294 \times 10^{-3} \\ 5 - 2048 \times 3.634 \times 10^{-3} \end{bmatrix} = \begin{bmatrix} 1.409 \\ -2.442 \end{bmatrix}$$

$$\therefore \quad \alpha_2 = \frac{1.409^2 + (-2.442)^2}{\begin{bmatrix} 1.409 & -2.442 \end{bmatrix} \begin{bmatrix} 1152 & 0 \\ 0 & 2048 \end{bmatrix} \begin{bmatrix} 1.409 \\ -2.442 \end{bmatrix}}$$

$$\therefore \quad \alpha_2 = \frac{1.409^2 + 2.442^2}{1152 \times 1.409^2 + 2048 \times 2.442^2} = 5.482 \times 10^{-4}$$

$$\therefore \quad \mathbf{x}_3 = \mathbf{x}_2 + \alpha_2 \mathbf{d}_2 = \begin{bmatrix} 6.294 \times 10^{-3} \\ 3.634 \times 10^{-3} \end{bmatrix} + 5.482 \times 10^{-4} \begin{bmatrix} 1.409 \\ -2.442 \end{bmatrix} = \begin{bmatrix} 7.066 \times 10^{-3} \\ 2.295 \times 10^{-3} \end{bmatrix}$$ [35%]

(e) The Steepest Descent Method is making good progress towards the minimum. By inspection the eigenvalues of the Hessian are 1152 and 2048. Therefore the convergence ratio β is bounded by:

$$\beta \leq \left[\frac{A-a}{A+a}\right]^2 = \left[\frac{2048-1152}{2048+1152}\right]^2 = 0.0784$$

This comparatively small value of β explains the good convergence.

The problem is quadratic so Newton's Method will converge in one iteration, and the Conjugate Gradient Method will converge in a number of iterations equal to the number of control variables, i.e. two iterations in this case. [15%]

Principal assessor's comments:
The least popular question but nevertheless done by 63% of candidates and well done by many of these. The most common sources of error were: a failure to check the second-order optimality conditions in part (b); and numerical slips in executing the Steepest Descent Method (SDM). Part (c), which asked for a standard derivation, was done surprisingly badly. Not many candidates recognised that they could straightforwardly calculate the upper bound on the SDM convergence ratio in part (e) to explain its performance.

Question 3

(a) The total cost of installation to be minimized is:

$$f(D,L) = 150D^2L + 25D^{2.5}L + 20DL$$

subject to
$$T = 15D^2L \geq 100$$
$$\therefore \quad g_1 = 100 - 15D^2L \leq 0$$

In principle, there are non-negativity bounds on D and L, but it is obvious that they cannot be active if g_1 is to be satisfied, and can therefore be omitted. [10%]

(b) Assuming the constraint on T is active, the problem is:

Minimize
$$f(D,L) = 150D^2L + 25D^{2.5}L + 20DL$$

subject to
$$h_1 = 100 - 15D^2L = 0$$

The Lagrangian is $\ell = 150D^2L + 25D^{2.5}L + 20DL + \lambda\left(100 - 15D^2L\right)$

$$\therefore \quad \frac{\partial \ell}{\partial D} = 300DL + 62.5D^{1.5}L + 20L - 30\lambda DL = 0 \qquad (1)$$

$$\frac{\partial \ell}{\partial L} = 150D^2 + 25D^{2.5} + 20D - 15D^2\lambda = 0 \qquad (2)$$

$$100 - 15D^2L = 0 \qquad (3)$$

From (1)
$$300D + 62.5D^{1.5} + 20 - 30\lambda D = 0 \qquad (4)$$

From (2)
$$15D^2\lambda = 150D^2 + 25D^{2.5} + 20D$$
$$\therefore \quad 30\lambda D = 300D + 50D^{1.5} + 40$$

Substituting in (4)
$$300D + 62.5D^{1.5} + 20 - 300D - 50D^{1.5} - 40 = 0$$
$$\therefore \quad 12.5D^{1.5} = 20 \quad \Rightarrow \quad D = 1.368 \text{ m}$$

From (3)
$$L = \frac{100}{15D^2}$$
$$\therefore \quad L = \frac{100}{15D^2} = \frac{100}{15(1.368)^2} = 3.562 \text{ m}$$
[40%]

(c) With the introduction of the new constraint we can no longer assume that the constraint on T is active, therefore we have now have a problem with two inequality constraints:

Minimize
$$f(D,L) = 150D^2L + 25D^{2.5}L + 20DL$$

Subject to
$$g_1 = 100 - 15D^2L \leq 0$$

and
$$g_2 = DL - 4 \leq 0$$

The Lagrangian is now:

$$\ell = 150D^2L + 25D^{2.5}L + 20DL + \mu_1\left(100 - 15D^2L\right) + \mu_2\left(DL - 4\right)$$

$$\therefore \quad \frac{\partial \ell}{\partial D} = 300DL + 62.5D^{1.5}L + 20L - 30\mu_1 DL + \mu_2 L = 0 \qquad (1)$$

$$\frac{\partial \ell}{\partial L} = 150D^2 + 25D^{2.5} + 20D - 15D^2\mu_1 + \mu_2 D = 0 \qquad (2)$$

$$\mu_1\left(100 - 15D^2 L\right) = 0 \qquad (3)$$

$$\mu_2\left(DL - 4\right) = 0 \qquad (4)$$

Case (i) $\mu_1 = 0$ and $\mu_2 = 0$

$(2) \Rightarrow \quad 150D^2 + 25D^{2.5} + 20D = 0$

$\therefore \quad D = 0$ (impossible)

or $\quad 150D + 25D^{1.5} + 20 = 0 \quad \Rightarrow$ no real non-negative solution for D

$\therefore$ **impossible**

Case (ii) $\mu_1 = 0$ and $\mu_2 > 0$

$(2) \Rightarrow \quad 150D^2 + 25D^{2.5} + 20D + \mu_2 D = 0$

$\therefore \quad D = 0$ (impossible)

or $\quad \mu_2 = -150D - 25D^{1.5} - 20 \quad \Rightarrow \quad \mu_2 < 0$

$\therefore$ **not a minimum**

Case (iii) $\mu_1 > 0$ and $\mu_2 = 0$

This is equivalent to the case solved in part (b) where the constraint on T is active and the constraint on A is inactive (in (b) it did not apply).

For this case we know that $D = 1.368$ m and $L = 3.562$ m.

We need to check that $g_2 = DL - 4 \le 0$ is not violated:

$g_2 = 1.368 \times 3.562 - 4 = 0.873 \quad \therefore \; g_2$ is violated

$\therefore$ **impossible**

Case (iv) $\mu_1 > 0$ and $\mu_2 > 0$

$(3) \Rightarrow \quad 100 - 15D^2 L = 0$

$\therefore \quad 15D^2 L = 100$

$(4) \Rightarrow \quad DL - 4 = 0 \quad \Rightarrow \quad L = \frac{4}{D}$

$\therefore \quad 15D^2 \frac{4}{D} = 100 \quad \Rightarrow \quad D = \frac{5}{3} \text{ m} \quad \Rightarrow \quad L = 4 \times \frac{3}{5} = 2.4 \text{ m}$

$(1) \Rightarrow \quad 300DL + 62.5D^{1.5}L + 20L - 30\mu_1 DL + \mu_2 L = 0$

$\therefore \quad 300D + 62.5D^{1.5} + 20 - 30\mu_1 D + \mu_2 = 0$

$$\therefore \quad 300\left(\frac{5}{3}\right)+62.5\left(\frac{5}{3}\right)^{1.5}+20-30\mu_1\left(\frac{5}{3}\right)+\mu_2=0$$

$$\therefore \quad 654.48-50\mu_1+\mu_2=0 \qquad (5)$$

$$(2)\Rightarrow \quad 150D^2+25D^{2.5}+20D-15D^2\mu_1+\mu_2 D=0$$

$$\therefore \quad 150D+25D^{1.5}+20-15D\mu_1+\mu_2=0$$

$$\therefore \quad 150\left(\frac{5}{3}\right)+25\left(\frac{5}{3}\right)^{1.5}+20-15\left(\frac{5}{3}\right)\mu_1+\mu_2=0$$

$$\therefore \quad 323.79-25\mu_1+\mu_2=0 \qquad (6)$$

$$(5)-(6)\Rightarrow \quad 330.69-25\mu_1=0 \quad \Rightarrow \quad \mu_1=13.228$$

$$\therefore \quad \mu_2=6.9$$

As $\mu_1>0$ and $\mu_2>0$ $\quad\therefore$ **a minimum**

Thus, the new optimal design is $D=1.667$ m and $L=2.4$ m. [50%]

Principal assessor's comments:

Most candidates showed that they had a good idea of how the Lagrange and Kuhn-Tucker multiplier methods work. Algebraic mistakes were the most common cause of failure to answer part (b) (about Lagrange multipliers). A number of candidates claimed dishonestly that the quoted solution solved their incorrect equations. This was penalised more harshly than answers where the candidate admitted that something must have gone wrong. Common problems in answering part (c) (about Kuhn-Tucker multipliers) were: not testing that the multipliers were positive at potential optima; not recognising that the solution to part (b) corresponded to one of the cases that needed testing; and failing to spot when cases could not correspond to a minimum because a multiplier would inevitably be negative for any physically plausible (i.e. non-negative) values of the control variables.

4 (a) A Markov chain is when, for each step in the process, $P(X_n = k)$ depends only on the value of X_{n-1} & not on previous values. For homogeneous case, for example, if

$$P(X_{n+1} = k \mid X_n = j) = P_{jk} = \text{elements of matrix } P$$

then

$$P(X_{n+2} = k \mid X_n = j) = (P^2)_{jk} \quad \text{etc}$$

[15%]

(b) $$P(X_{i+n} = k \mid X_i = j) = (P^n)_{jk}$$

$$= (P^s P^{n-s})_{jk} = \sum_l P^s_{jl} P^{n-s}_{lk}$$

$$= \sum_l P(X_{i+s} = l \mid X_i = j)\, P(X_{i+n} = k \mid X_{i+s} = l)$$

[15%]

(c) Denote a as 1 b as 2 c as 3

Then $P_{11} = P_{22} = P_{33} = \frac{1}{2}$

$P_{12} = \frac{1}{3}$ $P_{13} = \frac{1}{6}$ $P_{21} = P_{23} = P_{31} = P_{32} = \frac{1}{4}$

i.e. $$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

Let q_i = Expected no transitions till 1 if state $X_0 = i$

$\therefore$ $q_1 = E(\text{no after } 0 \text{ till } 1 \mid X_0 = 1, X_1 = 1)\, P(X_0 = 1, X_1 = 1)$

$+ E(\text{no} \ \text{"} \ \text{"} \ \text{"} \ 1 \mid X_0 = 1, X_1 = 2)\, P(X_0 = 1, X_1 = 2)$

$+ E(\text{no} \ \text{"} \ \text{"} \ \text{"} \ 1 \mid X_0 = 1, X_1 = 3)\, P(X_0 = 1, X_1 = 3)$

Now $E(\text{no after 0 till 1} \mid X_0=1, X_1=1) = 1$

& $E(\text{" " " " } 1 \mid X_1 = 2, X_1 = 1) = 1 + E(\text{no after 1 till 1} \mid X_1 = 2)$

$= 1 + E(\text{no after 0 till 1} \mid X_0 = 2)$

$= 1 + q_2$

$E(\text{" " " } 1 \mid X_1 = 3, X_1 = 1) = 1 + q_3$

$\therefore q_1 = P_{11} + (1+q_2)P_{12} + (1+q_3)P_{13}$

$= 1 + \frac{q_2}{3} + \frac{q_3}{6}$

Similarly $q_2 = P_{21} + (1+q_2)P_{22} + (1+q_3)P_{23}$

$= 1 + \frac{q_2}{2} + \frac{q_3}{4}$

Now by symmetry $q_3 = q_2$

$\therefore q_1 = 1 + \frac{q_2}{2}$ and $q_2 = 1 + \frac{3q_2}{4} \Rightarrow q_2 = 4$

[35%]

$\Rightarrow q_1 = 3$

(d) Final state has $[a\ b\ c]P = [a\ b\ c]$ (e-vector for e-value = 1)

$$\Rightarrow \begin{bmatrix} -\frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} \end{bmatrix}^T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \quad \Rightarrow \quad -\frac{a}{2} + \frac{b}{4} + \frac{c}{4} = 0$$

and $\frac{a}{3} - \frac{b}{2} + \frac{c}{4} = 0$

if $c = 1$ then $\left.\begin{array}{l} 4a - 2b = 2 \\ \frac{4a}{3} - 2b = -1 \end{array}\right\} \Rightarrow \frac{8a}{3} = 3 \Rightarrow a = \frac{9}{8} \quad b = \frac{5}{4} \quad c = 1$

Normalising s.t. $a+b+c=1$ Final proportions are $(.33, .37, .30)$

$\lambda = 1$ is largest e-value $\therefore$ This dominates after repeated multiplication by P so end state indept of starting state.

[35%]

Principal assessor's comments:

The most popular question and the best done. Part (b) which asked for a standard proof from lecture notes was omitted surprisingly often. The calculation of the expected number of transitions before returning to the initial state in part (c) was very well done, with simple numerical slips being the most common cause of errors. The calculation of the steady-state process distribution in part (d) was slightly less successful. Several candidates contrived to use the transpose of the transition matrix. A surprisingly large number could not normalise the (otherwise correct) proportions correctly. Few candidates discussed whether the steady-state distribution was actually reachable.

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Answers

Q1 (b) $\begin{bmatrix} \frac{1}{24} & \frac{1}{6} & -\frac{5}{24} \\ -\frac{5}{24} & \frac{1}{6} & \frac{1}{24} \end{bmatrix}$ (other equivalent results are possible)

Q2 (b) $x = 7.52 \times 10^{-3}$ m; $y = 2.44 \times 10^{-3}$ m

(d) $\mathbf{x}_2 = \begin{bmatrix} 6.294 \times 10^{-3} \\ 3.634 \times 10^{-3} \end{bmatrix}$; $\mathbf{x}_3 = \begin{bmatrix} 7.066 \times 10^{-3} \\ 2.295 \times 10^{-3} \end{bmatrix}$

Q3 (c) $D = \frac{5}{3}$ m; $L = \frac{12}{5}$ m

Q4 (c) 3

(d) $\left(\frac{9}{27} \quad \frac{10}{27} \quad \frac{8}{27} \right)$