ENGINEERING TRIPOS   PART IIA

Thursday 26 April 2012   9.00 to 12.00

Module 3A1

FLUID MECHANICS I

*Answer not more than **five** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments:

- *3A1 Data Sheet for Applications to External Flows (2 pages);*
- *Boundary Layer Theory Data Card (1 page);*
- *Incompressible Flow Data Card (2 pages).*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Version: Final version

1 (a) (i) Prove that the complex potential

$$F(z) = U\left(z + \frac{a^2}{z}\right)$$

where $z = x + iy$, describes an inviscid flow with upstream velocity U around a cylinder of radius a located at the origin. [10%]

(ii) Show that the tangential velocity component, u_θ, on the cylinder surface is related to the cartesian components (u, v) by

$$u_\theta = ie^{i\theta}(u - iv),$$

where θ is the polar angle measured anticlockwise from the x-axis. Hence, or otherwise, find an expression for u_θ. [10%]

(iii) A real fluid flow past a cylinder typically exhibits large-scale separation. What feature of the potential-flow solution suggests that such separation is to be expected? [10%]

(b) An engineer surmises that the separation may be eliminated by sucking fluid into a narrow pipe placed behind the cylinder on the x-axis. The pipe is represented by a line sink of strength m at a distance b from the origin. The sink and its associated images are shown in Fig. 1.

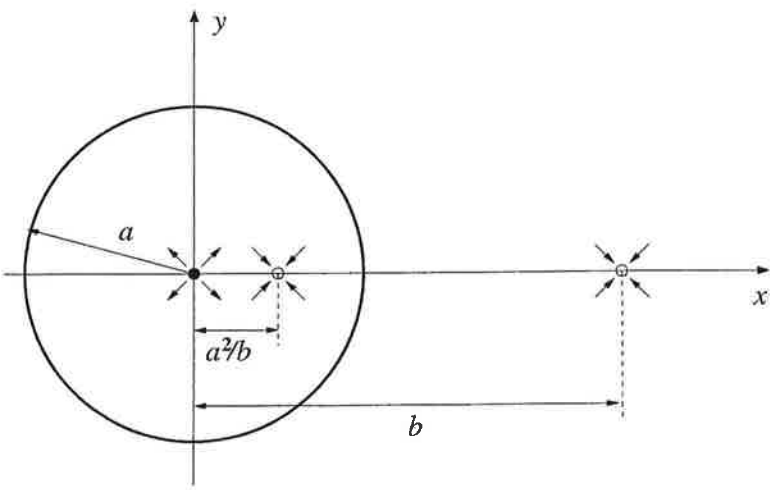


Fig. 1

(i) Show that the proposed image system maintains the cylinder surface as a streamline. [20%]

(ii) Find the *additional* surface velocity due to the presence of the suction pipe. [20%]

(iii) Comment on the likely success, or otherwise, of the engineer's idea. [15%]

(c) Discuss, qualitatively, alternative suction locations and their advantages or disadvantages relative to the configuration in (b). [15%]

2 The complex variable $\zeta = \xi + i\eta$ is defined in terms of $z = x + iy$ by the mapping

$$\zeta = z + \frac{a^2}{z} ,$$

where a is a constant.

(a) Show that the circle $z = R\,e^{i\beta}$, with $R > a$, maps on to the ellipse

$$\frac{\xi^2}{b_x^2} + \frac{\eta^2}{b_y^2} = 1 .$$

State explicitly the expressions for b_x and b_y . [20%]

(b) An inviscid fluid, with upstream velocity U in the x-direction, flows around the ellipse defined in (a). The flow speed on the surface of the ellipse is denoted by q. Find q^2, in terms of U , a , R and β . You may assume without proof that the complex potential for the corresponding flow around the cylinder in the z-plane is $U(z + R^2/z)$. [40%]

(c) In a real flow, q represents the flow speed outside a surface boundary layer. The initial, laminar, development of the boundary layer can be calculated using Thwaites' method, according to which the momentum thickness θ is given by

$$[\theta(s)]^2 = \frac{0.45\nu}{[q(s)]^6} \int_0^s [q(s')]^5 ds' ,$$

where ν is the fluid's kinematic viscosity and s the distance around the ellipse from the front stagnation point. [40%]

(i) By writing $\beta = \pi - \varepsilon$, with $\varepsilon \ll 1$, find the leading-order approximations for s and q .

(ii) Hence find an expression for θ^2 in the region of the front stagnation point.

(iii) Comment briefly on any notable features of your expression.

3 (a) Show that a three-dimensional doublet, with strength $\mu = 2\pi a^3 U$, located at $x = x_d, y = 0, z = 0$ can be used to represent the potential flow around a sphere of radius a , centred at that location and moving with speed U in the x-direction through a stationary fluid. [20%]

(b) Consider the case where U is a constant, independent of time. [40%]

(i) Explain carefully why the flow field is unsteady in the absolute frame.

(ii) Find an expression for the surface pressure field in terms of U , a , the angle θ from the doublet axis, and the pressure p_0 at $\theta = 0$. You may ignore the gravitational component.

(iii) Comment on the fore-aft symmetry of the pressure field, and hence on the associated horizontal force.

(c) The sphere speed U now varies with time. [40%]

(i) Find the additional pressure-field component that is due to the unsteadiness in U .

(ii) What is the associated horizontal force on the sphere?

(iii) How would you interpret this force in terms of an effective modification of a physical property of the sphere?

4 Figure 2 shows the development of a laminar boundary layer along a flat plate, where δ is the thickness of the boundary layer, L the length of development and U the freestream velocity. We consider here steady high Reynolds number flows.

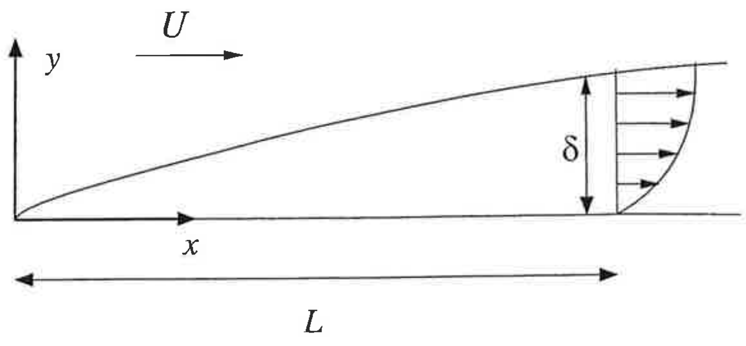


Fig. 2

(a) How does the magnitude of δ compare qualitatively to L ? [10%]

(b) Within the boundary layer, what are the orders of magnitude of the x-velocity component u and its derivatives $\partial u/\partial x$ and $\partial u/\partial y$? Explain your reasoning. [10%]

(c) Consider the continuity equation for the boundary layer. Deduce the order of magnitude of the y-velocity component v . [20%]

(d) Consider the y-direction momentum equation. Use Prandtl's order of magnitude argument to deduce an equation for the pressure. Hence explain why the inviscid flow theory is successful in predicting the lift force on an airfoil. [30%]

(e) Apply Prandtl's order of magnitude argument to the x-direction momentum equation for the boundary layer. Explain carefully why

$$\frac{\delta}{L} \sim \sqrt{\frac{1}{Re}}, \qquad Re = \frac{UL}{\nu},$$

where ν is the kinematic viscosity of the fluid. Deduce the approximate momentum equation for the boundary layer. [30%]

5 (a) Thwaites defined two dimensionless parameters l and m for laminar boundary layers, such that

$$l = \frac{\theta}{U}\left(\frac{\partial u}{\partial y}\right)_w = \frac{\theta}{U}\frac{\tau_w}{\mu}, \qquad m = \frac{\theta^2}{U}\left(\frac{\partial^2 u}{\partial y^2}\right)_w$$

where $U = U(x)$ is the freestream velocity, μ the fluid's dynamic viscosity, τ_w the wall shear stress and θ the momentum thickness of the boundary layer.

(i) Show that [20%]

$$m = -\frac{\theta^2}{\nu}\frac{dU}{dx}$$

where ν is the fluid's kinematic viscosity.

(ii) Use the momentum integral equation to deduce [20%]

$$U\frac{d(\theta^2)}{dx} = 2\nu[(H+2)m + l]$$

where H is the shape factor.

(iii) Assuming that

$$2[(H+2)m + l] = 0.45 + 6m\ ,$$

deduce that

$$[\theta(x)]^2 = \frac{0.45\nu}{[U(x)]^6}\int_0^x U(x')^5 dx'\ ,$$

where x is the location of interest. You may assume that either $U(0) = 0$ or $\theta(0) = 0$. [20%]

(b) (i) Consider a linearly decelerating flow with a freestream distribution

$$U(x) = U_0\left(1 - \frac{x}{L}\right),$$

where L and U_0 are constants. Use the above formulae to calculate θ and m. [20%]

(ii) Assuming that the separation point is given by $m = 0.09$, find its location and the pressure coefficient there, using the pressure and velocity at $x = 0$ as the reference quantities. [20%]

6 (a) Consider classical two-dimensional thin aerofoil theory. A camber line is given by

$$y_c = h\frac{x}{c}\left(1-\frac{x}{c}\right)\left(K-\frac{x}{c}\right)$$

where c is the chord length, h and K are constants. Calculate the g coefficients as defined in the 3A1 Data Sheet for Applications to External Flows. [30%]

(b) Hence find an expression for the local aerofoil loading (defined as the local pressure coefficient difference between the lower and upper surfaces of the aerofoil) at zero incidence. [20%]

(c) From your expression in (b) identify the singular (flat-plate-like) contribution to the loading. Sketch this and the remainder (for $K = 1$) on a plot of pressure coefficient C_p versus x/c. Also sketch a typical contribution to the distribution from the thickness. Show how the overall pressure coefficient distribution is derived. [20%]

(d) Find the value of K such that the maximum load due to the finite camber component is at the aerofoil quarter-chord point. [30%]

7 (a) Consider two-dimensional aerofoil flow and write brief notes accompanied by appropriate sketches on the following topics :

(i) boundary layers and separation; [20%]

(ii) separation bubbles; [10%]

(iii) stall mechanisms; [20%]

(iv) remedial measures for stall. [10%]

(b) Consider now three-dimensional wings and write brief notes accompanied by appropriate sketches on the following topics :

(i) the role of the spanwise wing loading distribution on stalling behaviour; [20%]

(ii) boundary layer cross-flow on swept wings and the impact on stall. [20%]

8 Figure 3 shows the outline of a saloon car.

(a) Sketch the approximate distribution of the surface pressure coefficient along the centre-line running from A to B along the upper surface. Make sure you identify the key points. [25%]

(b) Highlight possible areas of flow separation and comment on their significance to drag generation. [25%]

(c) Suggest modifications to the body shape between A and B to address the problems identified in (b). [25%]

(d) Apart from the upper surface, make two additional suggestions for reducing the aerodynamic drag and explain briefly why they might be effective. [25%]

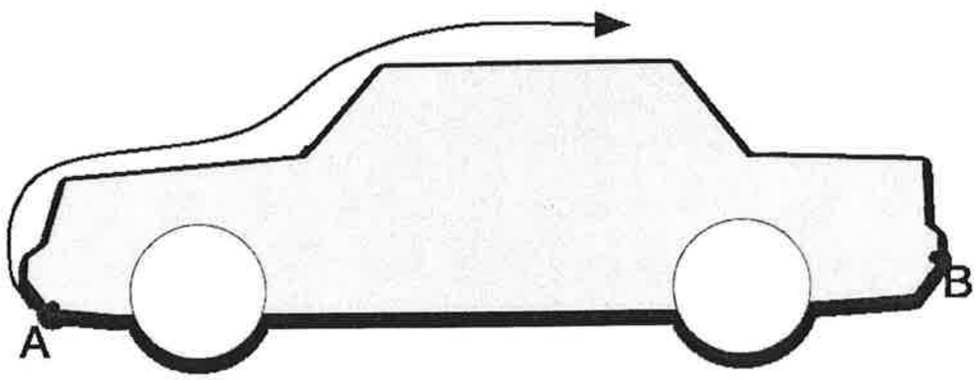


Fig. 3

END OF PAPER

Version: Final version

3A1 Data Sheet for Applications to External Flows

Aerodynamic Coefficients

For a flow with free-stream density, ρ, velocity U and pressure p_∞:

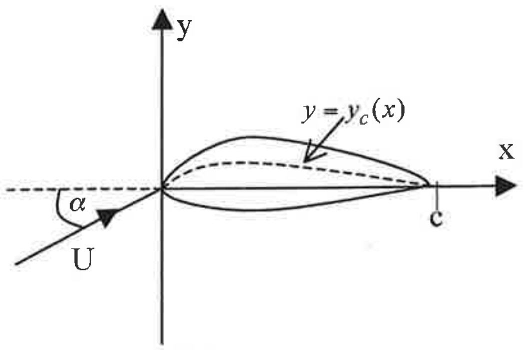
Pressure coefficient: $$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2}$$

Section lift and drag coefficients: $$c_l = \frac{\text{lift}\,(N/m)}{\frac{1}{2}\rho U^2 c},\quad c_d = \frac{\text{drag}\,(N/m)}{\frac{1}{2}\rho U^2 c}$$ (section chord c)

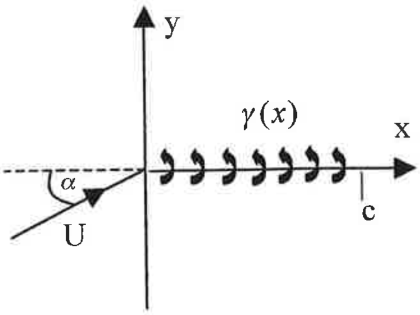
Wing lift and drag coefficients: $$C_L = \frac{\text{lift}\,(N)}{\frac{1}{2}\rho U^2 S},\quad C_D = \frac{\text{drag}\,(N)}{\frac{1}{2}\rho U^2 S}$$ (wing area S)

Thin Aerofoil Theory

Geometry



Approximate representation



Pressure coefficient: $c_p = \pm\gamma / U$

Pitching moment coefficient: $c_m = (\text{moment about } x = 0) / \frac{1}{2}\rho U^2 c^2$

Coordinate transformation: $x = c(1 + \cos\theta)/2 = c\cos^2(\theta/2)$

Incidence solution: $$\gamma = -2U\alpha \frac{1 - \cos\theta}{\sin\theta},\quad c_l = 2\pi\alpha,\quad c_m = c_l/4$$

Camber solution: $$\gamma = -U\left[g_0 \frac{1 - \cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} g_n \sin n\theta\right],\ \text{where}$$

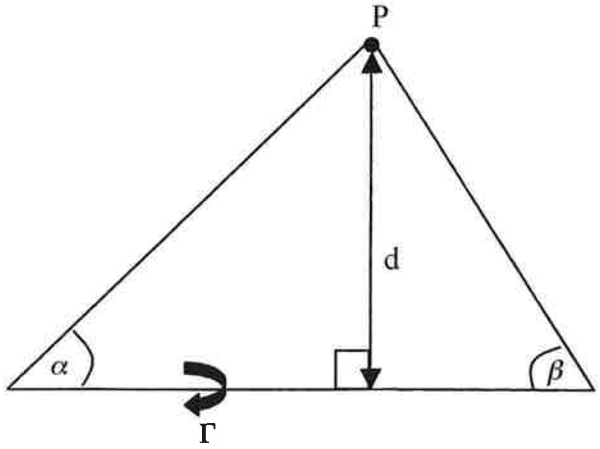
$$g_0 = \frac{1}{\pi}\int_0^{\pi}\left(-2\frac{dy_c}{dx}\right)d\theta,\quad g_n = \frac{2}{\pi}\int_0^{\pi}\left(-2\frac{dy_c}{dx}\right)\cos n\theta\, d\theta$$

$$c_l = \pi\left(g_0 + \frac{g_1}{2}\right),\quad c_m = \frac{\pi}{4}\left(g_0 + g_1 + \frac{g_2}{2}\right) = \frac{c_l}{4} + \frac{\pi}{8}(g_1 + g_2)$$

Glauert Integral

$$\int_{0}^{\pi} \frac{\cos n\phi}{\cos\phi - \cos\theta} d\phi = \pi \frac{\sin n\theta}{\sin\theta}$$

Line Vortices



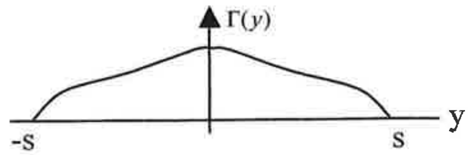
A straight element of circulation Γ induces a velocity at P of

$$\frac{\Gamma}{4\pi d}(\cos\alpha + \cos\beta)$$

perpendicular to the plane containing P and the element.

Lifting-Line Theory

Spanwise circulation distribution:



Aspect ratio: $A_R = 4s^2 / S$

Wing lift: $L = \rho U \int_{-s}^{s} \Gamma(y) dy$

Downwash angle: $\alpha_d(y) = \frac{1}{4\pi U} \int_{-s}^{s} \frac{d\Gamma(\eta)/d\eta}{y - \eta} d\eta$

Induced drag: $D_i = \rho U \int_{-s}^{s} \Gamma(y) \alpha_d(y) dy$

Fourier series for circulation: $\Gamma(y) = Us \sum_{n \text{ odd}} G_n \sin n\theta$, with $y = -s\cos\theta$

Relation between lift and induced drag:

$$C_{Di} = (1+\delta)\frac{C_L^2}{\pi A_R}, \text{ where } \delta = 3\left(\frac{G_3}{G_1}\right)^2 + 5\left(\frac{G_5}{G_1}\right)^2 + \ldots$$

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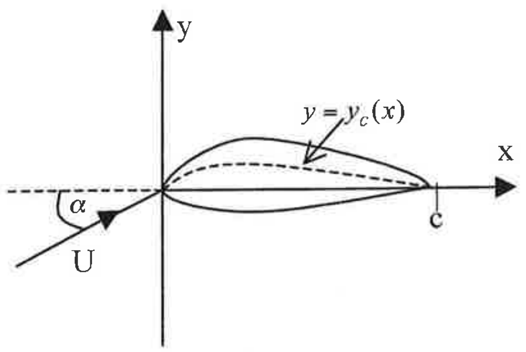
Thin Aerofoil Theory

Geometry



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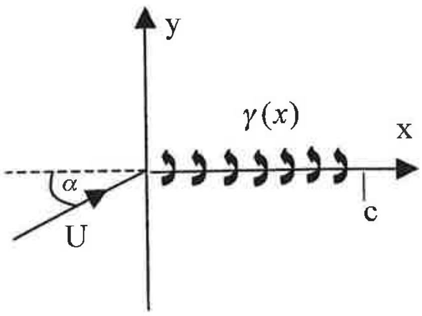

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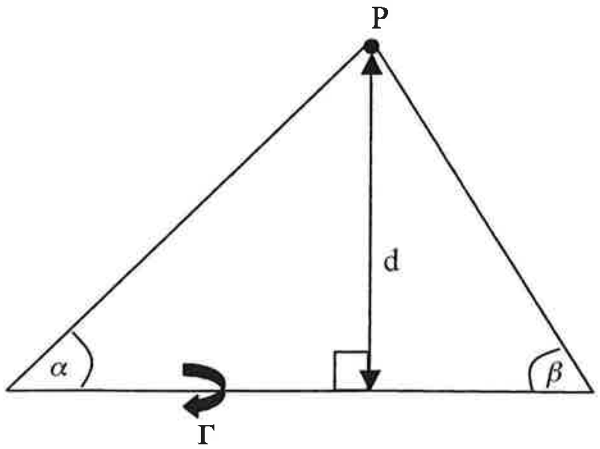

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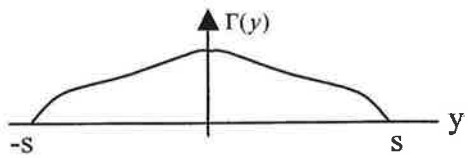

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Module 3A1
Boundary Layer Theory Data Card

Displacement thickness;

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_1}\right) dy$$

Momentum thickness;

$$\theta = \int_0^\infty \frac{(U_1 - u)u}{U_1^2} dy = \int_0^\infty \left(1 - \frac{u}{U_1}\right) \frac{u}{U_1} dy$$

Energy thickness;

$$\delta_E = \int_0^\infty \frac{(U_1^2 - u^2)u}{U_1^3} dy = \int_0^\infty \left(1 - \left(\frac{u}{U_1}\right)^2\right) \frac{u}{U_1} dy$$

$$H = \frac{\delta^*}{\theta}$$

Prandtl's boundary layer equations (laminar flow);

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{dp_1}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned}$$

von Karman momentum integral equation;

$$\frac{d\theta}{dx} + \frac{H+2}{U_1}\theta\frac{dU_1}{dx} = \frac{\tau_o}{\rho U_1^2} = \frac{C_f'}{2}$$

Boundary layer equations for turbulent flow;

$$\begin{aligned} \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} &= \frac{-1}{\rho}\frac{d\overline{p}}{dx} - \frac{\partial \overline{u'v'}}{\partial y} + \nu\frac{\partial^2 \overline{u}}{\partial y^2} \\ \frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} &= 0 \end{aligned}$$

Module 3A1: Fluid Mechanics I

INCOMPRESSIBLE FLOW DATA CARD

Continuity equation $\nabla \cdot \boldsymbol{u} = 0$

Momentum equation (inviscid) $\rho \dfrac{D\boldsymbol{u}}{Dt} = -\nabla p + \rho \boldsymbol{g}$

D/Dt denotes the material derivative, $\partial/\partial t + \boldsymbol{u} \cdot \nabla$

Vorticity $\boldsymbol{\omega} = \text{curl } \boldsymbol{u}$

Vorticity equation (inviscid) $\dfrac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u}$

Kelvin's circulation theorem (inviscid) $\dfrac{D\Gamma}{Dt} = 0, \ \Gamma = \oint \boldsymbol{u} \cdot d\boldsymbol{l} = \int \boldsymbol{\omega} \cdot d\boldsymbol{S}$

For an irrotational flow

velocity potential ϕ $\boldsymbol{u} = \nabla \phi$ and $\nabla^2 \phi = 0$

Bernoulli's equation for inviscid flow: $\dfrac{p}{\rho} + \tfrac{1}{2}|\boldsymbol{u}|^2 + gz + \dfrac{\partial \phi}{\partial t} = \text{constant}$ throughout flow field

TWO-DIMENSIONAL FLOW

Streamfunction ψ

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad u_\theta = -\frac{\partial \psi}{\partial r}$$

Lift force Lift / unit length $= \rho U(-\Gamma)$

For an irrotational flow

complex potential *F(z)* $F(z) = \phi + i\psi$ is a function of $z = x + iy$

$$\frac{dF}{dz} = u - iv$$

Summary of simple 2 - D flow fields

	ϕ	ψ	$F(z)$	$\boldsymbol{u}$
Uniform flow (x - wise)	Ux	Uy	Uz	$u = U,\ v = 0$
Source at origin	$\frac{m}{2\pi}\ln r$	$\frac{m}{2\pi}\theta$	$\frac{m}{2\pi}\ln z$	$u_r = \frac{m}{2\pi r},\ u_\theta = 0$
Doublet (x - wise) at origin	$-\frac{\mu\cos\theta}{2\pi r}$	$\frac{\mu\sin\theta}{2\pi r}$	$-\frac{\mu}{2\pi z}$	$u_r = \frac{\mu\cos\theta}{2\pi r^2},\ u_\theta = \frac{\mu\sin\theta}{2\pi r^2}$
Vortex at origin	$\frac{\Gamma}{2\pi}\theta$	$-\frac{\Gamma}{2\pi}\ln r$	$-\frac{i\Gamma}{2\pi}\ln z$	$u_r = 0,\ u_\theta = \frac{\Gamma}{2\pi r}$

THREE-DIMENSIONAL FLOW

Summary of simple 3 - D flow fields

	ϕ	$\boldsymbol{u}$
Source at origin	$-\frac{m}{4\pi r}$	$u_r = \frac{m}{4\pi r^2},\quad u_\theta = 0,\quad u_\psi = 0$
Doublet at origin (with θ the angle from the doublet axis)	$-\frac{\mu\cos\theta}{4\pi r^2}$	$u_r = \frac{\mu\cos\theta}{2\pi r^3},\quad u_\theta = \frac{\mu\sin\theta}{4\pi r^3},\quad u_\psi = 0$