ENGINEERING TRIPOS PART IIA

Friday 4 May 2012 9 to 10.30

Module 3C5

DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
Datasheet: 3C5 Dynamics and 3C6 Vibration (6 pages)

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

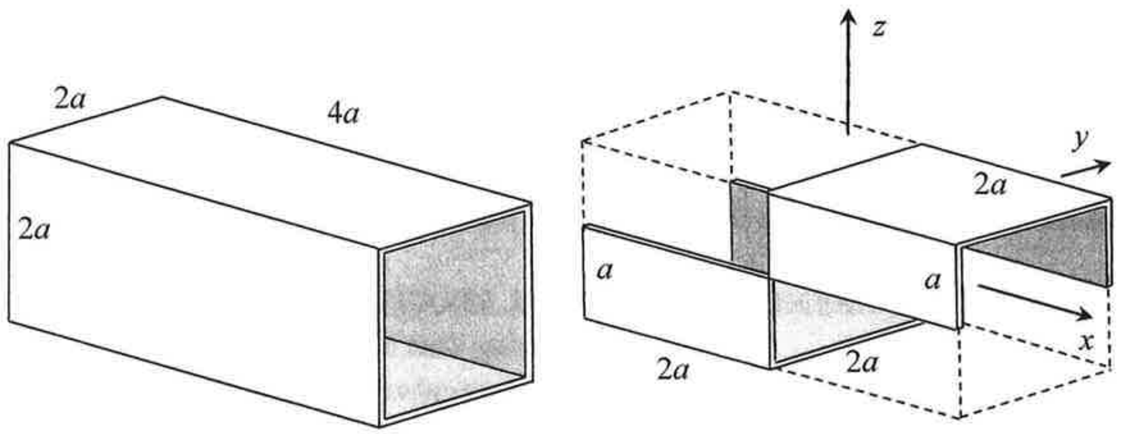
You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Final version

1 (a) Figure 1a shows a thin-walled tube of square section with length $4a$, side $2a$ and mass m. Find the principal moments of inertia of the tube centred on G, the centre of mass. [20%]

(b) Figure 1b shows the same tube with two opposite quarters removed (shown dashed). The centre of mass is in the same location as before, and the mass is now $m/2$. Find the inertia matrix for the modified tube centred on G relative to the axes aligned with x,y,z as shown. [60%]

(c) The modified tube from part (b) is forced to spin about the x axis with angular velocity Ω. Find the necessary couple (in magnitude and direction). [20%]



(a) (b)

Fig. 1

2 A thin coin of mass m and radius a wobbles on a horizontal table as shown in Fig. 2. The motion may be assumed to be in steady state with the centre of the coin G at rest. The coin is in contact with the table at point P where it rolls without slip. The motion of the coin is described by Euler angles θ, ϕ, ψ. Orthogonal unit vectors **i**, **j**, **k** (not fixed in the body) are defined in the figure, with **j** horizontal to complete the right-handed triad.

(a) Draw a free-body diagram of the coin and hence find a vector expression for the couple acting on the coin. [10%]

(b) Find $\dot{\phi}$, the steady rate of wobbling of the coin. [50%]

(c) Find an expression for the rate of turning of the head of the coin. Discuss what happens if (i) θ is small, and (ii) θ is close to $\pi/2$. [40%]

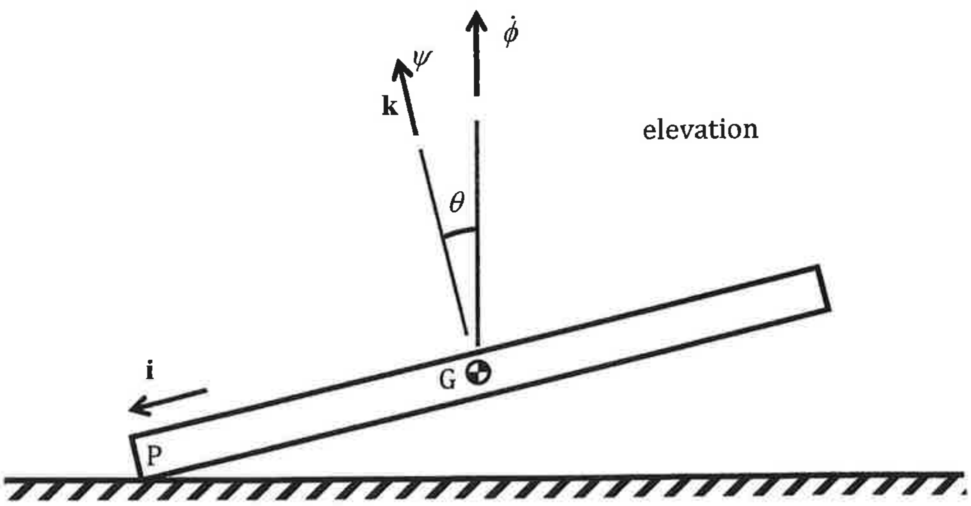


Fig. 2

3 (a) (i) Describe with the aid of sketches the principles of operation of a strap-down inertial navigation system. Your discussion should include the integration requirements for rate gyro and accelerometer signals, and the means by which g-computer compensation is used. [40%]

(ii) Explain the features of a locally-level platform. What are the implications regarding mechanical versus computational complexity, when compared to a strap-down system? [10%]

(b) A rigid body has a body-fixed axis system (x, y, z) aligned with the principal axes of inertia, with corresponding moments of inertia (A, B, C). These axes are initially aligned with a fixed global axis system (X, Y, Z). The body moves in such a way that its subsequent orientation can be described by two Euler angles: a rotation θ taken about the Y axis, and a rotation ψ then taken about the new orientation of the z axis.

(i) Show that the rotation rates expressed in the body-fixed axes are given by

$$\omega_1 = \dot{\theta}\sin\psi$$
$$\omega_2 = \dot{\theta}\cos\psi$$
$$\omega_3 = \dot{\psi}$$

[25%]

(ii) By considering the Lagrange equation for ψ, derive the Euler equation of motion associated with ω_3 for the case of free motion with no applied couple. [25%]

4 Two uniform rigid rods are pivoted at one end about a fixed horizontal axis, and connected by a linear spring of stiffness k at the other end, as shown in Fig. 3. The rods have length L and masses M_1 and M_2, and their motions are described by angles $\theta_1(t)$ and $\theta_2(t)$ to the vertical. The spring has zero natural length. The rods are closely adjacent on the pivot axis and the spring always lies in the plane of the figure.

(a) By using Lagrange's equation, show that the equation of motion of the second rod is

$$\frac{1}{3}M_2L^2\ddot{\theta}_2 + \frac{1}{2}M_2gL\sin\theta_2 + kL^2\sin(\theta_2 - \theta_1) = 0$$

where g is the acceleration due to gravity, and similarly derive the equation of motion of the first rod. [35%]

(b) Assuming that $\theta_1(t)$ and $\theta_2(t)$ are both small, find the mass and stiffness matrices of the system. Hence find expressions for the natural frequencies associated with small-amplitude vibration. [35%]

(c) Now suppose that the two rods lie in exactly the same plane and cannot pass through each other. Instead, they will collide and bounce. Contact may be assumed to happen at a small stud at the end of the beams, near the spring position. Suggest how the contact force might be represented to give a model of this process, and show how this force can be expressed in generalized forces that would appear in the Lagrange equations. [30%]

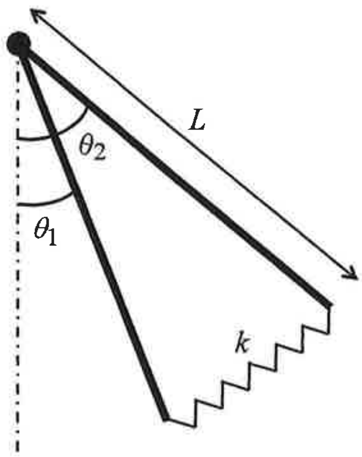


Fig. 3

END OF PAPER

Final version

Part IIA Data sheet
Module 3C5 Dynamics
Module 3C6 Vibration

DYNAMICS IN THREE DIMENSIONS

Axes fixed in direction

(a) Linear momentum for a general collection of particles m_i:

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}^{(e)}$$

where $\boldsymbol{p} = M\,\boldsymbol{v}_G$, M is the total mass, $\boldsymbol{v}_G$ is the velocity of the centre of mass and $\boldsymbol{F}^{(e)}$ the total external force applied to the system.

(b) Moment of momentum about a general point P

$$\boldsymbol{Q}^{(e)} = (\boldsymbol{r}_G - \boldsymbol{r}_P) \times \dot{\boldsymbol{p}} + \dot{\boldsymbol{h}}_G$$
$$= \dot{\boldsymbol{h}}_P + \dot{\boldsymbol{r}}_P \times \boldsymbol{p}$$

where $\boldsymbol{Q}^{(e)}$ is the total moment of external forces about P. Here, $\boldsymbol{h}_P$ and $\boldsymbol{h}_G$ are the moments of momentum about P and G respectively, so that for example

$$\boldsymbol{h}_P = \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_P) \times m_i \dot{\boldsymbol{r}}_i$$
$$= \boldsymbol{h}_G + (\boldsymbol{r}_G - \boldsymbol{r}_P) \times \boldsymbol{p}$$

where the summation is over all the mass particles making up the system.

(c) For a rigid body rotating with angular velocity $\boldsymbol{\omega}$ about a fixed point P at the origin of coordinates

$$\boldsymbol{h}_P = \int \boldsymbol{r} \times (\boldsymbol{\omega} \times \mathbf{r})\,dm = I\,\boldsymbol{\omega}$$

where the integral is taken over the volume of the body, and where

$$I = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}, \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \qquad \boldsymbol{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and $A = \int (y^2 + z^2)\,dm$ $\quad B = \int (z^2 + x^2)\,dm$ $\quad C = \int (x^2 + y^2)\,dm$

$D = \int yz\,dm$ $\quad E = \int zx\,dm$ $\quad F = \int xy\,dm$

where all integrals are taken over the volume of the body.

Axes rotating with angular velocity $\boldsymbol{\Omega}$

Time derivatives of vectors must be replaced by the "rotating frame" form, so that for example

$$\dot{\boldsymbol{p}} + \boldsymbol{\Omega} \times \boldsymbol{p} = \boldsymbol{F}^{(e)}$$

where the time derivative is evaluated in the moving reference frame.

When the rate of change of the position vector $\boldsymbol{r}$ is needed, as in 1(b) above, it is usually easiest to calculate velocity components directly in the required directions of the axes. Application of the general formula needs an extra term unless the origin of the frame is fixed.

Euler's dynamic equations (governing the angular motion of a rigid body)

(a) Body-fixed reference frame:

$$A\,\dot{\omega}_1 - (B - C)\,\omega_2\,\omega_3 = Q_1$$
$$B\,\dot{\omega}_2 - (C - A)\,\omega_3\,\omega_1 = Q_2$$
$$C\,\dot{\omega}_3 - (A - B)\,\omega_1\,\omega_2 = Q_3$$

where A, B and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes aligned with the principal axes of inertia of the body at P.

(b) Non-body-fixed reference frame for axisymmetric bodies (the "Gyroscope equations"):

$$A\,\dot{\Omega}_1 - (A\,\Omega_3 - C\,\omega_3)\,\Omega_2 = Q_1$$
$$A\,\dot{\Omega}_2 + (A\,\Omega_3 - C\,\omega_3)\,\Omega_1 = Q_2$$
$$C\,\dot{\omega}_3 = Q_3$$

where A, A and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes such that ω_3 and Q_3 are aligned with the symmetry axis of the body. The reference frame (not fixed in the body) rotates with angular velocity $\boldsymbol{\Omega} = [\Omega_1, \Omega_2, \Omega_3]$ with $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$.

Lagrange's equations

For a holonomic system with generalised coordinates q_i

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where T is the total kinetic energy, V is the total potential energy, and Q_i are the non-conservative generalised forces.

VIBRATION MODES AND RESPONSE

Discrete systems

1. The forced vibration of an N-degree-of-freedom system with mass matrix M and stiffness matrix K (both symmetric and positive definite) is

$$M\,\underline{\ddot{y}} + K\,\underline{y} = \underline{f}$$

where $\underline{y}$ is the vector of generalised displacements and $\underline{f}$ is the vector of generalised forces.

2. Kinetic energy

$$T = \frac{1}{2}\underline{\dot{y}}^t M \underline{\dot{y}}$$

Potential energy

$$V = \frac{1}{2}\underline{y}^t K \underline{y}$$

3. The natural frequencies ω_n and corresponding mode shape vectors $\underline{u}^{(n)}$ satisfy

$$K\underline{u}^{(n)} = \omega_n{}^2 M\underline{u}^{(n)} \; .$$

4. Orthogonality and normalisation

$$\underline{u}^{(j)^t} M \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

$$\underline{u}^{(j)^t} K \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ \omega_n^2, & j = k \end{cases}$$

5. General response

The general response of the system can be written as a sum of modal responses

$$\underline{y}(t) = \sum_{j=1}^{N} q_j(t)\,\underline{u}^{(j)} = U\underline{q}(t)$$

where U is a matrix whose N columns are the normalised eigenvectors $\underline{u}^{(j)}$ and q_j can be thought of as the "quantity" of the jth mode.

Continuous systems

The forced vibration of a continuous system is determined by solving a partial differential equation: see p. 6 for examples.

$$T = \frac{1}{2}\int \dot{u}^2 dm$$

where the integral is with respect to mass (similar to moments and products of inertia).

See p. 4 for examples.

The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see p. 4) and boundary conditions, assuming harmonic time dependence.

$$\int u_j(x)\,u_k(x)\,dm = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

The general response of the system can be written as a sum of modal responses

$$w(x,t) = \sum_j q_j(t)\,u_j(x)$$

where $w(x,t)$ is the displacement and q_j can be thought of as the "quantity" of the jth mode.

6. Modal coordinates $\underline{q}$ satisfy

$$\ddot{\underline{q}} + \left[\text{diag}(\omega_j^2)\right]\underline{q} = \underline{Q}$$

where $\underline{y} = U\underline{q}$ and the modal force vector

$$\underline{Q} = U^t \underline{f} \ .$$

Each modal amplitude $q_j(t)$ satisfies

$$\ddot{q}_j + \omega_j^2 q_j = Q_j$$

where $Q_j = \int f(x,t)\, u_j(x)\, dm$ and $f(x,t)$ is the external applied force distribution.

7. Frequency response function

For input generalised force f_j at frequency ω and measured generalised displacement y_k the transfer function is

$$H(j,k,\omega) = \frac{y_k}{f_j} = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(j,k,\omega) = \frac{y_k}{f_j} \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

For force F at frequency ω applied at point x, and displacement w measured at point y, the transfer function is

$$H(x,y,\omega) = \frac{w}{F} = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(x,y,\omega) = \frac{w}{F} \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2 + 2i\,\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

8. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)} u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with low modal overlap, if the factor $u_n(x)\, u_n(y)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

9. Impulse response

For a unit impulsive generalised force $f_j = \delta(t)$ the measured response y_k is given by

$$g(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin\omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin\omega_n t\, e^{-\omega_n\zeta_n t}$$

for $t \geq 0$ (with small damping).

For a unit impulse applied at $t = 0$ at point x, the response at point y is

$$g(x,y,t) = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin\omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(x,y,t) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin\omega_n t\, e^{-\omega_n\zeta_n t}$$

for $t \geq 0$ (with small damping).

10. Step response

For a unit step generalised force $f_j = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$ the measured response y_k is given by

$$h(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - \cos\omega_n t\right]$$

for $t \geq 0$ (with no damping), or

$$h(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - \cos\omega_n t \, e^{-\omega_n \zeta_n t}\right]$$

for $t \geq 0$ (with small damping).

For a unit step force applied at $t = 0$ at point x, the response at point y is

$$h(x,y,t) = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2} \left[1 - \cos\omega_n t\right]$$

for $t \geq 0$ (with no damping), or

$$h(t) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2} \left[1 - \cos\omega_n t \, e^{-\omega_n \zeta_n t}\right]$$

for $t \geq 0$ (with small damping).

Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is $\dfrac{V}{\tilde{T}} = \dfrac{\underline{y}^t K \underline{y}}{\underline{y}^t M \underline{y}}$ where $\underline{y}$ is the vector of generalised coordinates, M is the mass matrix and K is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions on p. 6.

If this quantity is evaluated with any vector $\underline{y}$, the result will be

(1) $\geq$ the smallest squared frequency;

(2) $\leq$ the largest squared frequency;

(3) a good approximation to ω_k^2 if $\underline{y}$ is an approximation to $\underline{u}^{(k)}$.

(Formally, $\dfrac{V}{\tilde{T}}$ is *stationary* near each mode.)

GOVERNING EQUATIONS FOR CONTINUOUS SYSTEMS

Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $w(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\dfrac{\partial^2 w}{\partial t^2} - P\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}P\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}m\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $f(x,t)$ per unit length.
Polar moment of area is $J = (\pi/2)(a^4 - b^4)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\dfrac{\partial^2 \theta}{\partial t^2} - GJ\dfrac{\partial^2 \theta}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}GJ\int\left(\dfrac{\partial \theta}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho J\int\left(\dfrac{\partial \theta}{\partial t}\right)^2 dx$

Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $w(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 w}{\partial t^2} - EA\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}EA\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $w(x,t)$, applied transverse force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 w}{\partial t^2} + EI\dfrac{\partial^4 w}{\partial x^4} = f(x,t)$	$V = \dfrac{1}{2}EI\int\left(\dfrac{\partial^2 w}{\partial x^2}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.

3C5 2012 Answers

1 (a) $\frac{4}{3}ma^2,\quad 2ma^2,\quad 2ma^2$

(b) $ma^2\begin{bmatrix} 2/3 & 0 & -3/8 \\ 0 & 1 & 0 \\ -3/8 & 0 & 1 \end{bmatrix}$

(c) $\frac{3}{8}ma^2\Omega^2\mathbf{j}$

2 (a) $-mga\cos\theta\,\mathbf{j}$

(b) $\dot{\phi} = 2\sqrt{\frac{g}{a\sin\theta}}$

(c) (i) $\sqrt{\frac{\theta^3 g}{a}}$ relative to a frame *fixed in the table*

(ii) $-2\sqrt{\frac{g}{a}}\left(\frac{\pi}{2}-\theta\right)$ relative to a frame in which *contact point with table stays fixed*

4 (a) $\frac{1}{3}M_1L^2\ddot{\theta}_1 + \frac{1}{2}M_1gL\sin\theta_1 - kL^2\sin(\theta_2-\theta_1) = 0$

(b) $M = \frac{L^2}{3}\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix},\quad K = \begin{bmatrix} \frac{1}{2}M_1gL + kL^2 & -kL^2 \\ -kL^2 & \frac{1}{2}M_2gL + kL^2 \end{bmatrix}$

$\omega^2 = \frac{3g}{2L},\quad \frac{kL^2(M_1+M_2)+gLM_1M_2/2}{M_1M_2L^2/3}$