ENGINEERING TRIPOS PART IIA

Wednesday 9 May 2012 9.00 to 10.30

Module 3C7

MECHANICS OF SOLIDS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
Special datasheet: Module 3C7 Mechanics of Solids (2 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Final version

1 Consider the Airy stress function

$$\phi(r,\theta) = \sigma_0\left(-\frac{a^2 \ln r}{2} + \frac{r^2}{4}\right) + \sigma_0\left(-\frac{r^2}{4} - \frac{a^4}{4r^2} + \frac{a^2}{2}\right)\cos 2\theta$$

which satisfies $\nabla^4\phi = 0$.

(a) Derive the stress components σ_{rr}, $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$ from ϕ. [30%]

(b) Show that ϕ describes the elastic solution for the stress components around a small hole of radius a in a large thin sheet that remotely carries a uniaxial stress σ_0. [30%]

(c) A closed-end thin walled cylindrical pressure vessel of diameter D and wall thickness t carries an internal pressure p. The cylinder has a small hole of radius a in its wall remote from the ends of the cylinder. Find an expression for the distribution of hoop stress along the dashed line in Fig. 1. This line is parallel to the axis of the cylinder and passes through the centre of the hole. You may assume that $t \ll a \ll D$. [40%]

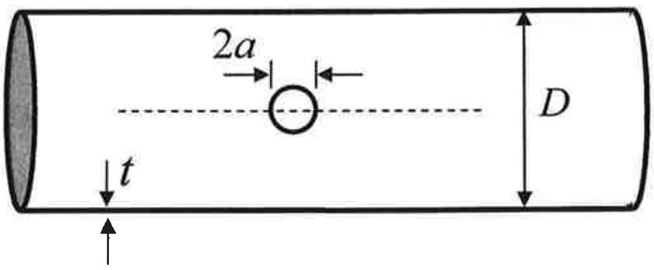

Fig. 1

2 A long shaft has a uniform circular cross-section of diameter D. It is made from a material with shear modulus G, and carries a torque T that causes a twist per unit length β.

(a) The shear stress in the shaft at a radius r is given by $\tau = G\beta r$. By direct integration, find the torque T carried by the shaft. [20%]

(b) Find a Prandtl stress function ψ for this problem and show that:

(i) ψ satisfies the governing equation $\nabla^2\psi = -2G\beta$ for elastic torsion. [20%]

(ii) ψ satisfies the equilibrium equation $T = 2\int_A \psi \, dA$, where A is the cross-sectional area of the shaft. [30%]

(c) It is proposed that a rectangular slot be cut in the shaft, as shown in the cross-sectional view of the shaft in Fig. 2. Describe the consequences of this for (i) the stiffness T / β and (ii) the peak stress in the shaft. [30%]

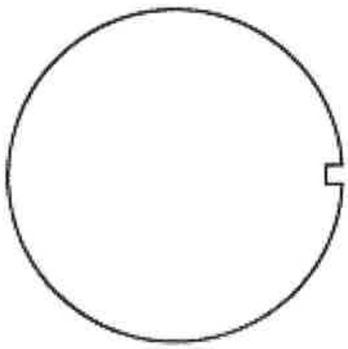

Fig. 2

(TURN OVER

3 (a) Explain briefly the differences between Lower and Upper bound plasticity solutions. [30%]

(b) Figure 3 shows a plane strain extrusion process in which a billet is extruded sideways from a container as the punch moves from right to left. The material of the billet can be considered to be rigid-perfectly plastic with a flow stress in shear of magnitude k. The thickness of the billet is t as is the width of the slot from which the material is extruded. Velocity discontinuities for an Upper Bound analysis are indicated by dashed lines and the material emerges in a direction ψ controlled by the position of point P. Point P is located by a distance xt from the right edge of the slot as shown in Fig. 3, where x is an unknown to be determined from the Upper Bound analysis.

(i) If the interface between the deforming material and the interior of the container is well lubricated explain why $\psi = \tan^{-1} 2$. [20%]

(ii) In practice, the interface between the deforming material and the container offers an interfacial friction stress equal to fk where $0 < f < 1$. If $f = 0.2$ what would be the expected value of x ? [30%]

(iii) If $f = 0.4$ make an estimate of the extrusion force F required per unit width of material in terms of k and t. [20%]

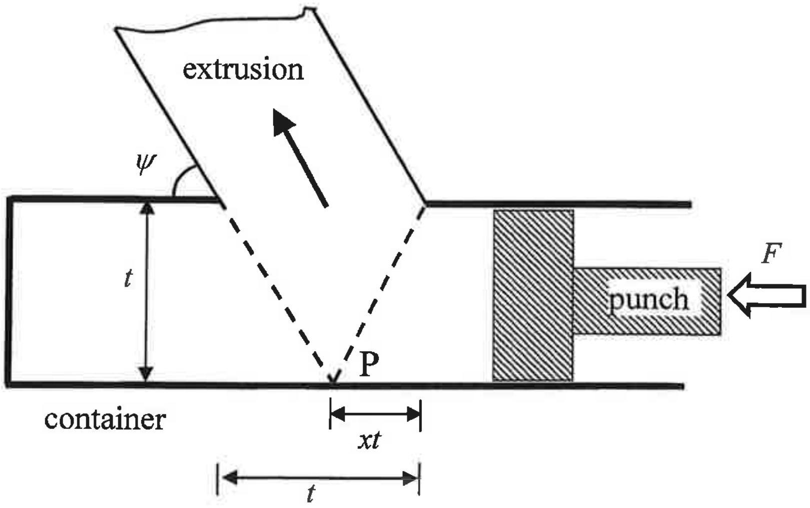

Fig. 3

4 A thin walled tube of mean radius r and wall thickness t has closed ends and can be subjected to an internal pressure p. The tube material is elastic, perfectly plastic, with Young's modulus E, Poisson's ratio $\nu = 0.3$ and yield stress in tension Y. This closed tube is held in a stiff test machine whose grips are fixed so that the tube cannot lengthen.

(a) In a first test, Fig. 4a, the internal pressure is increased until the tube yields. Determine an expression for the yield pressure p in terms of Y, r and t using the von Mises criterion. [30%]

(b) In a second test, Fig. 4b, the internal pressure is maintained at one half of the yield pressure of part (a) but an increasing torque T is applied. Again using the von Mises criterion, determine the value of T at which the tube yields in terms of Y, r and t. [40%]

(c) Calculate the direction of plastic straining in the first test. Express your answer in terms of the ratios of the plastic strain increments. [30%]

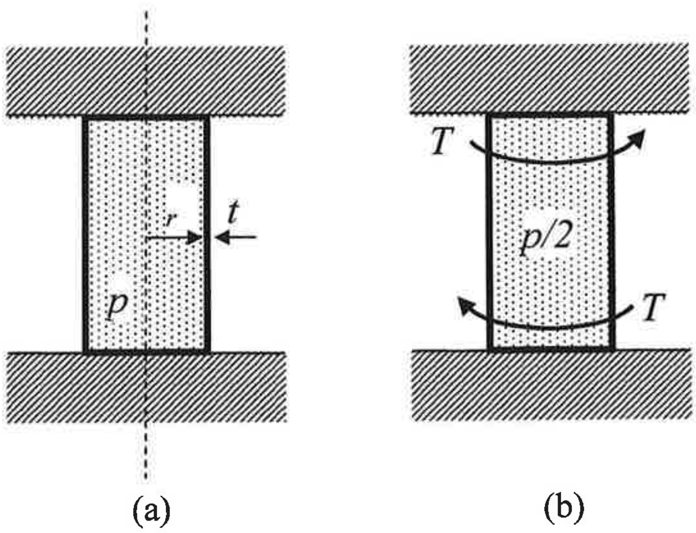

Fig. 4

END OF PAPER

Module 3C7: Mechanics of Solids

ELASTICITY and PLASTICITY FORMULAE

1. Axi-symmetric deformation : discs, tubes and spheres

	Discs and tubes	Spheres
Equilibrium	$\sigma_{\theta\theta} = \dfrac{d(r\sigma_{rr})}{dr} + \rho\omega^2 r^2$	$\sigma_{\theta\theta} = \dfrac{1}{2r}\dfrac{d(r^2\sigma_{rr})}{dr}$
Lamé's equations (in elasticity)	$\sigma_{rr} = A - \dfrac{B}{r^2} - \dfrac{3+\nu}{8}\rho\omega^2 r^2 - \dfrac{E\alpha}{r^2}\int_c^r rT\,dr$	$\sigma_{rr} = A - \dfrac{B}{r^3}$
	$\sigma_{\theta\theta} = A + \dfrac{B}{r^2} - \dfrac{1+3\nu}{8}\rho\omega^2 r^2 + \dfrac{E\alpha}{r^2}\int_c^r rT\,dr - E\alpha T$	$\sigma_{\theta\theta} = A + \dfrac{B}{2r^3}$

2. Plane stress and plane strain

	Cartesian coordinates	Polar coordinates
Strains	$\varepsilon_{xx} = \dfrac{\partial u}{\partial x}$	$\varepsilon_{rr} = \dfrac{\partial u}{\partial r}$
	$\varepsilon_{yy} = \dfrac{\partial v}{\partial y}$	$\varepsilon_{\theta\theta} = \dfrac{u}{r} + \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$
	$\gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$	$\gamma_{r\theta} = \dfrac{\partial v}{\partial r} + \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r}$
Compatibility	$\dfrac{\partial^2\gamma_{xy}}{\partial x\partial y} = \dfrac{\partial^2\varepsilon_{xx}}{\partial y^2} + \dfrac{\partial^2\varepsilon_{yy}}{\partial x^2}$	$\dfrac{\partial}{\partial r}\left\{r\dfrac{\partial\gamma_{r\theta}}{\partial\theta}\right\} = \dfrac{\partial}{\partial r}\left\{r^2\dfrac{\partial\varepsilon_{\theta\theta}}{\partial r}\right\} - r\dfrac{\partial\varepsilon_{rr}}{\partial r} + \dfrac{\partial^2\varepsilon_{rr}}{\partial\theta^2}$
or (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}(\sigma_{xx} + \sigma_{yy}) = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right\}(\sigma_{rr} + \sigma_{\theta\theta}) = 0$
Equilibrium	$\dfrac{\partial\sigma_{xx}}{\partial x} + \dfrac{\partial\sigma_{xy}}{\partial y} = 0$	$\dfrac{\partial}{\partial r}(r\sigma_{rr}) + \dfrac{\partial\sigma_{r\theta}}{\partial\theta} - \sigma_{\theta\theta} = 0$
	$\dfrac{\partial\sigma_{yy}}{\partial y} + \dfrac{\partial\sigma_{xy}}{\partial x} = 0$	$\dfrac{\partial\sigma_{\theta\theta}}{\partial\theta} + \dfrac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0$
$\nabla^4\phi = 0$ (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}\left\{\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2}\right\} = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right\} \times \left\{\dfrac{\partial^2\phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}\right\} = 0$
Airy Stress Function	$\sigma_{xx} = \dfrac{\partial^2\phi}{\partial y^2}$	$\sigma_{rr} = \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}$
	$\sigma_{yy} = \dfrac{\partial^2\phi}{\partial x^2}$	$\sigma_{\theta\theta} = \dfrac{\partial^2\phi}{\partial r^2}$
	$\sigma_{xy} = -\dfrac{\partial^2\phi}{\partial x\partial y}$	$\sigma_{r\theta} = -\dfrac{\partial}{\partial r}\left\{\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}\right\}$

3. Torsion of prismatic bars

Prandtl stress function: $\sigma_{zx}\ (= \tau_x) = \dfrac{\partial \psi}{\partial y}, \quad \sigma_{zy}\ (= \tau_y) = -\dfrac{\partial \psi}{\partial x}$

Equilibrium: $T = 2 \int_A \psi \, dA$

Governing equation for elastic torsion: $\nabla^2 \psi = -2G\beta$ where β is the angle of twist per unit length.

4. Total potential energy of a body

$$\Pi = U - W$$

where $U = \frac{1}{2} \int_V \underset{\sim}{\varepsilon}^T [D] \underset{\sim}{\varepsilon} \, dV$, $W = \underset{\sim}{P}^T \underset{\sim}{u}$ and $[D]$ is the elastic stiffness matrix.

5. Principal stresses and stress invariants

Values of the principal stresses, σ_P, can be obtained from the equation

$$\begin{vmatrix} \sigma_{xx} - \sigma_P & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \sigma_P & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \sigma_P \end{vmatrix} = 0$$

This is equivalent to a cubic equation whose roots are the values of the 3 principal stresses, i.e. the possible values of σ_P.

Expanding: $\sigma_P^3 - I_1 \sigma_P^2 + I_2 \sigma_P - I_3 = 0$ where $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$,

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} \quad \text{and} \quad I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}.$$

6. Equivalent stress and strain

Equivalent stress $\bar{\sigma} = \sqrt{\frac{1}{2}} \left\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right\}^{1/2}$

Equivalent strain increment $d\bar{\varepsilon} = \sqrt{\frac{2}{3}} \left\{d\varepsilon_1^2 + d\varepsilon_2^2 + d\varepsilon_3^2\right\}^{1/2}$

7. Yield criteria and flow rules

Tresca

Material yields when maximum value of $|\sigma_1 - \sigma_2|$, $|\sigma_2 - \sigma_3|$ or $|\sigma_3 - \sigma_1|$ $= Y = 2k$, and then,

if σ_3 is the intermediate stress, $d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = \lambda(1 : -1 : 0)$ where $\lambda \neq 0$.

von Mises

Material yields when, $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2 = 6k^2$, and then

$$\frac{d\varepsilon_1}{\sigma'_1} = \frac{d\varepsilon_2}{\sigma'_2} = \frac{d\varepsilon_3}{\sigma'_3} = \frac{d\varepsilon_1 - d\varepsilon_2}{\sigma_1 - \sigma_2} = \frac{d\varepsilon_2 - d\varepsilon_3}{\sigma_2 - \sigma_3} = \frac{d\varepsilon_3 - d\varepsilon_1}{\sigma_3 - \sigma_1} = \lambda = \frac{3}{2}\frac{d\bar{\varepsilon}}{\bar{\sigma}}.$$

Answers to 3C7: Mechanics of Solids (2011-2012)

1. (c) $\frac{pD}{8t}\left[4+\frac{3a^2}{r^2}+\frac{5a^4}{r^4}\right]$

2. (a) $T=\frac{\pi G\beta D^4}{32}$

   (b) $\psi = G\beta\left[\left(\frac{D}{2}\right)^2 - r^2\right]$

3. (b)(ii) $x=0.45$

   (b)(iii) $F=2.68kt$

4. (a) $p=1.125Yt/r$

   (b) $T=3.14r^2tY$

   (c) $d\varepsilon_h^p : d\varepsilon_a^p : d\varepsilon_t^p = -4.25:1:3.25$