ENGINEERING TRIPOS PART IIA

Thursday 3 May 2012 2.30 to 4

Module 3D7

FINITE ELEMENT METHODS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: 3D7 Data Sheet (3 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Version: FINAL

1 The strong governing equation for a beam on an elastic foundation reads

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right)+\alpha v=f$$

where EI is the bending stiffness of the beam, v is the deflection, α is the stiffness of the foundation and f is a distributed load.

(a) For the configuration shown in Fig. 1, derive a suitable weak form of the equation. [30%]

(b) For a finite element formulation of this problem, comment on the required continuity of the shape functions and explain what can happen if the shape functions do not possess sufficient continuity. [20%]

(c) For a two-node element that runs from x_1 to x_2, with $x_1 < x_2$, compute the k_{11} component of the element stiffness matrix for the case $\alpha = 0$. Assume that the first degree of freedom corresponds to the deflection of the beam and that EI is constant. [50%]

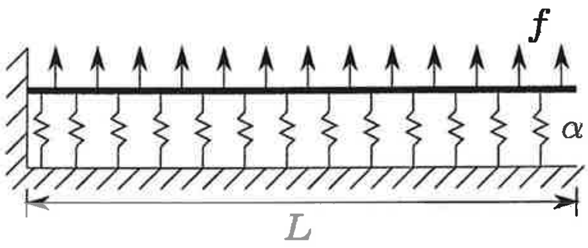


Fig. 1

2 Consider the element shown in Fig. 2.

(a) Determine the shape functions of the element. [10%]

(b) Compute the strain components ε_{xx}, ε_{yy} and ε_{xy} in terms of the nodal displacements. [30%]

(c) Compute the stiffness matrix of the element for a plane-stress linear elastic problem, taking into account the boundary conditions indicated in Fig. 2. [30%]

(d) Assuming constant mass density, compute the consistent mass matrix of the element. [30%]

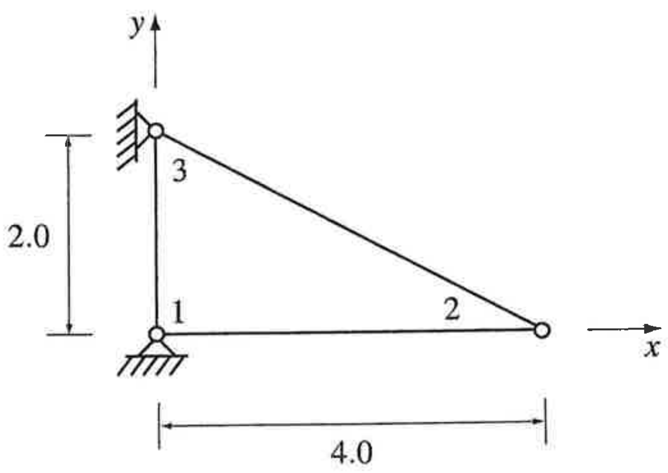


Fig. 2

3 On a domain Ω with boundary Γ, a modified heat equation in two dimensions reads

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+c_x\frac{\partial T}{\partial x}+c_y\frac{\partial T}{\partial y}+s=0$$

where T is the temperature, s is a heat source term, and k, c_x and c_y are constants. The temperature boundary condition is

$$T=0 \quad \text{on } \Gamma$$

(a) Show that a weak form of this modified heat equation is

$$k\int_\Omega\left(\frac{\partial w_0}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial w_0}{\partial y}\frac{\partial T}{\partial y}\right)d\Omega-c_x\int_\Omega w_0\frac{\partial T}{\partial x}\,d\Omega-c_y\int_\Omega w_0\frac{\partial T}{\partial y}\,d\Omega=\int_\Omega w_0 s\,d\Omega$$

where w_0 is a test function. [30%]

(b) For the quadrilateral four-node element shown in Fig. 3, discretise the terms below and evaluate the $(1,2)$ component of the resulting matrix.

$$-c_x\int_\Omega w_0\frac{\partial T}{\partial x}\,d\Omega-c_y\int_\Omega w_0\frac{\partial T}{\partial y}\,d\Omega$$

[50%]

(c) If numerical integration is used to evaluate the integrals in (b), give the recommended minimum number of Gauss quadrature points for four-node and nine-node elements. Justify your answers. [20%]

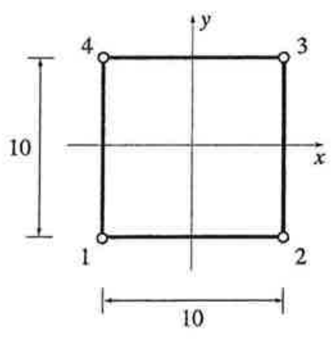


Fig. 3

4 The critical time step for explicit time-stepping schemes depends on the largest eigenvalue of the generalised eigenvalue problem

$$\boldsymbol{K}\boldsymbol{\psi}^{(i)} = \lambda^{(i)}\boldsymbol{M}\boldsymbol{\psi}^{(i)} \tag{1}$$

where $\boldsymbol{K}$ is the global stiffness matrix, $\boldsymbol{M}$ is the global mass matrix, $\lambda^{(i)}$ is the ith eigenvalue and $\boldsymbol{\psi}^{(i)}$ is the ith eigenvector.

(a) For a single elastic bar element with two nodes, constant stiffness EA and a consistent mass matrix, compute the largest eigenvalue as a function of the element length l. Comment on how the largest eigenvalue varies with EA. [60%]

(b) In finite element simulations, it is safe to compute the critical time step by considering the eigenvalues of element-wise problems rather than the global problem. Explain why the local eigenvalues provide a safe estimate. [10%]

(c) For the model systems $\dot{y}^{(i)} + \lambda^{(i)} y^{(i)} = 0$, where $\lambda^{(i)}$ is the ith eigenvalue of Eq. (1), compute the critical time step for the forward Euler method. Comment on the suitability of the forward Euler method for this type of problem. Recall that for the equation $\dot{y} = f(y,t)$, the forward Euler method is given by

$$y_{n+1} = y_n + \Delta t f(y_n, t_n)$$

[30%]

END OF PAPER

Engineering Tripos Part IIA
Module 3D7: Finite Element Methods

Data Sheet

Element relationships

Elasticity

Displacement	$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{a}_e$
Strain	$\boldsymbol{\epsilon} = \boldsymbol{B}\boldsymbol{a}_e$
Stress (2D/3D)	$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\epsilon}$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B}\, dV$
Element force vector (body force only)	$\boldsymbol{f}_e = \int_{V_e} \boldsymbol{N}^T \boldsymbol{f}\, dV$

Heat conduction

Temperature	$T = \boldsymbol{N}\boldsymbol{a}_e$
Temperature gradient	$\nabla T = \boldsymbol{B}\boldsymbol{a}_e$
Element conductance matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B}\, dV$

Beam bending

Displacement	$v = \boldsymbol{N}\boldsymbol{a}_e$
Curvature	$\kappa = \boldsymbol{B}\boldsymbol{a}_e$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T EI \boldsymbol{B}\, dV$

Elasticity matrices

2D plane strain

$$\boldsymbol{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$

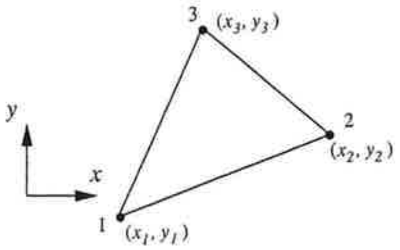
2D plane stress

$$\boldsymbol{D} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

Heat conductivity matrix (2D)

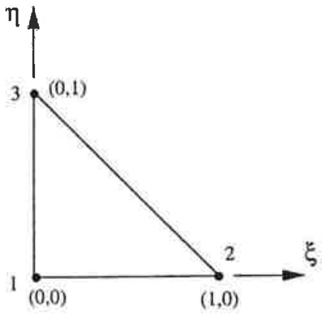
$$\boldsymbol{D} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

Shape functions

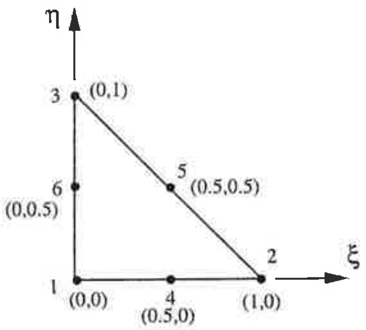


$$N_1 = \left(\left(x_2y_3 - x_3y_2\right) + \left(y_2 - y_3\right)x + \left(x_3 - x_2\right)y\right)/2A$$
$$N_2 = \left(\left(x_3y_1 - x_1y_3\right) + \left(y_3 - y_1\right)x + \left(x_1 - x_3\right)y\right)/2A$$
$$N_3 = \left(\left(x_1y_2 - x_2y_1\right) + \left(y_1 - y_2\right)x + \left(x_2 - x_1\right)y\right)/2A$$

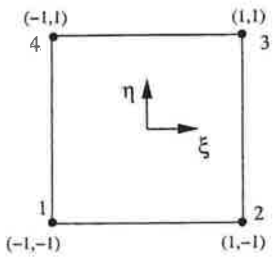
A = area of triangle



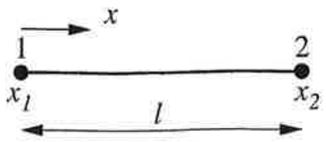
$$N_1 = 1 - \xi - \eta$$
$$N_2 = \xi$$
$$N_3 = \eta$$



$$N_1 = 2\left(1 - \xi - \eta\right)^2 - \left(1 - \xi - \eta\right)$$
$$N_2 = 2\xi^2 - \xi$$
$$N_3 = 2\eta^2 - \eta$$
$$N_4 = 4\xi\left(1 - \xi - \eta\right)$$
$$N_5 = 4\eta\xi$$
$$N_6 = 4\eta\left(1 - \xi - \eta\right)$$



$$N_1 = \left(1 - \xi\right)\left(1 - \eta\right)/4$$
$$N_2 = \left(1 + \xi\right)\left(1 - \eta\right)/4$$
$$N_3 = \left(1 + \xi\right)\left(1 + \eta\right)/4$$
$$N_4 = \left(1 - \xi\right)\left(1 + \eta\right)/4$$



Hermitian element

$$N_1 = \frac{-(x-x_2)^2(-l+2(x_1-x))}{l^3}$$

$$M_1 = \frac{(x-x_1)(x-x_2)^2}{l^2}$$

$$N_2 = \frac{(x-x_1)^2(l+2(x_2-x))}{l^3}$$

$$M_2 = \frac{(x-x_1)^2(x-x_2)}{l^2}$$

Gauss integration in one dimension on the domain $(-1,1)$

Using n Gauss integration points, a polynomial of degree $2n-1$ is integrated exactly.

number of points n	location ξ_i	weight w_i
1	0	2
2	$-\frac{1}{\sqrt{3}}$	1
	$\frac{1}{\sqrt{3}}$	1
3	$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
	0	$\frac{8}{9}$
	$\sqrt{\frac{3}{5}}$	$\frac{5}{9}$