ENGINEERING TRIPOS PART IIA

Tuesday 1 May 2012 9 to 10.30

Module 3F1

SIGNALS AND SYSTEMS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

FINAL version

1 Consider the discrete-time system given by the following difference equation

$$y_{k+2} = y_{k+1} - \alpha y_k + x_k$$

where α is a constant, x_k is the input and y_k is the output.

(a) Find the transfer function from $X(z)$ to $Y(z)$ as a function of α. [10%]

(b) For each $\alpha = 0$, $\alpha = 1/4$, $\alpha = 1/2$ and $\alpha = 2$, find the poles of the system and conclude whether it is stable or unstable. [20%]

(c) For each $\alpha = 0$, $\alpha = 1/4$ and $\alpha = 1/2$, calculate and sketch the pulse response of the system for all $k \geq 0$. [30%]

Hint: The data book and some of the following z-transforms may be helpful:

$$p^k \leftrightarrow \frac{1}{1 - pz^{-1}}, \quad k \leftrightarrow \frac{z^{-1}}{(1 - z^{-1})^2}, \quad kp^k \leftrightarrow \frac{pz^{-1}}{(1 - pz^{-1})^2}$$

(d) A controller is added to control the motion of the system, so that the following equation holds:

$$X(z) = K(R(z) - Y(z))$$

where $R(z)$ is the reference signal and K is a constant.

(i) Calculate the closed-loop transfer function from $R(z)$ to $Y(z)$ in terms of α and K. [15%]

(ii) For each $\alpha = 0$, $\alpha = 1/4$, $\alpha = 1/2$ and $\alpha = 2$, find a K that stabilises each closed loop system. Does there exist a single K that stabilises all four closed loop systems? [25%]

2 (a) In a discrete-time linear system:

(i) State the definition of stability for the system. [10%]

(ii) Using the definition of stability, show that the system

$$G(z) = \frac{1}{z-1}$$

is unstable. [30%]

Hint: Some of the following z-transforms may be helpful:

$$p^k \leftrightarrow \frac{1}{1-pz^{-1}}, \quad k \leftrightarrow \frac{z^{-1}}{(1-z^{-1})^2}, \quad kp^k \leftrightarrow \frac{pz^{-1}}{(1-pz^{-1})^2}$$

(b) Two independent random variables, X_1 and X_2, have pdfs $f_1(x)$ and $f_2(x)$. A new process Y is formed from the sum of X_1 and X_2.

(i) Show that the conditional pdf of Y, given that $X_1 = x_1$, is given by

$$f(y|x_1) = f_2(y - x_1)$$

and hence obtain an expression for the pdf of Y, $f_Y(y)$, in terms of $f_1(x)$ and $f_2(x)$. [20%]

(ii) The characteristic function of a random variable X is given by

$$\Phi_X(u) = E[e^{juX}]$$

where $E[.]$ denotes expectation. Obtain an expression for $\Phi_Y(u)$ in terms of Φ_{X_1} and Φ_{X_2} for the above process Y. [20%]

(iii) Explain why characteristic functions are helpful in probability analysis. [20%]

3 (a) How does the power spectral density (PSD) of an ergodic random signal relate to its auto-correlation function (ACF)? [10%]

(b) The ACF of an ergodic random signal X is

$$r_{XX}(\tau) = \begin{cases} P\left(1 - \frac{|\tau|}{T_0}\right) & \text{if } |\tau| \leq T_0 \\ 0 & \text{otherwise} \end{cases}$$

Obtain an expression for $S_X(\omega)$, the PSD of X. [20%]

(c) The signal X is passed through a linear system with frequency response $H(\omega)$ to produce a random output signal Y. Calculate $S_Y(\omega)$, the PSD of Y. [20%]

(d) If $r_{XX}(\tau)$ is known and $r_{YY}(\tau)$ can be measured in a large safety-critical linear system, arranged as in part (c), explain what aspects of $H(\omega)$ may be computed for the system while it is operating normally, and how this can be done. [30%]

(e) If the cross-correlation function $r_{XY}(\tau)$ of the above system can also be measured during normal operation, discuss how an improved estimate of $H(\omega)$ may be obtained. [20%]

4 (a) Consider the following three lists of binary codeword lengths:

(i) 1,2,2,3

(ii) 2,2,2,3

(iii) 1,2,3,3

For each of the three lists:

(i) Does there exist a prefix-free code with this codeword length? [15%]

(ii) And if so, could this be a Huffman code? [15%]

(b) Let two binary random variables X and Y be such that:

$$P_X(0) = p, \quad P_{Y|X}(1|1) = 1, \quad P_{Y|X}(1|0) = \frac{1}{2}$$

(i) Express as functions of p and in units of bits, the following entropies and mutual information:

$$H(X), \quad H(Y|X), \quad H(XY), \quad H(Y), \quad H(X|Y), \quad I(X;Y)$$

[30%]

(ii) Describe the channel between X and Y as a matrix or in graphical form. [15%]

(iii) What is the capacity of this channel? [25%]

Note: In part (b), you may express all your answers using the binary entropy function

$$h(x) = -x\log_b(x) - (1-x)\log_b(1-x)$$

You may also need its derivative

$$h'(x) = \log_b\left(\frac{1-x}{x}\right)$$

where the logarithmic base b must match the base that you chose for entropy in your calculations.

END OF PAPER

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Short answers

1. (a) $\dfrac{1}{z^2-z+\alpha}$;
   (b) $z=0$, $z=1$, unstable; $z=\frac{1}{2}, z=\frac{1}{2}$, stable;
   $z=\frac{1\pm i}{2}$, stable; $z=\frac{1\pm i\sqrt{7}}{2}$, unstable.
   (c) $y_k=\{0,0,1,1,1,\cdots\}$;
   $y_k=0$ for $k=0,1$, $y_k=2(1-k)(1/2)^{k-1}$ for $k\geq 2$;
   $y_k=0$ for $k=0,1$, $y_k=(1/\sqrt{2})^{k-3}\sin(\frac{\pi}{4}(k-1))$ for $k\geq 2$.
   (d) (i) $\dfrac{K}{z^2-z+\alpha+K}$;
   (ii) $0<K<1$; $-\frac{1}{4}<K<\frac{3}{4}$; $-\frac{1}{2}<K<\frac{1}{2}$; $-2<K<-1$; No.

2. (b) (i) $f_Y(y)=\int_{-\infty}^{\infty} f_2(y-x_1)\, f_1(x_1)\, dx$ (convolution)
   (ii) $\Phi_Y(u)=\Phi_{X_2}(u)\,\Phi_{X_1}(u)$

3. (b) $S_X(\omega)=P\,T_0\,\text{sinc}^2\left(\dfrac{\omega T_0}{2}\right)=\dfrac{2P}{T_0}\left(\dfrac{1-\cos(\omega T_0)}{\omega^2}\right)$
   (c) $S_Y(\omega)=|H(\omega)|^2\,P\,T_0\,\text{sinc}^2\left(\dfrac{\omega T_0}{2}\right)$
   (d) $|H(\omega)|=\sqrt{\dfrac{S_Y(\omega)}{S_X(\omega)}}$
   (e) $H(\omega)=\dfrac{S_{XY}(\omega)}{S_X(\omega)}$

4. (a) (i) no, yes, yes; (ii) Huffman only for third code.
   (b) (i) $H(X)=h(p)$; $H(Y|X)=p$; $H(XY)=h(p)+p$; $H(Y)=h(p/2)$;
   $H(X|Y)=h\left(\frac{p}{2-p}\right).(1-\frac{p}{2})=h(p)+p-h(p/2)$
   $I(X;Y)=h(p/2)-p$
   (iii) $p_{\max}=0.4$; $I_{\max}(X;Y)=0.3219$ bit per use.