ENGINEERING TRIPOS PART IIA

Monday 7 May 2012 9 to 10.30

Module 3F2

SYSTEMS AND CONTROL

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Version: final

1 (a) Define the *matrix exponential* function e^M, where M is a square matrix. [10%]

(b) Show that, if A is a square matrix, then [10%]

$$\frac{d}{dt}\left(e^{-At}\right) = -Ae^{-At} = -e^{-At}A$$

(c) Describe one way of evaluating e^{At}. [10%]

(d) Show that the solution of $\dot{x} = Ax + Bu$ is [30%]

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

(e) If $\dot{x} = Ax + Bu$, with

$$A = \begin{bmatrix} -1 & 0 \\ 2 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$u(t)$ is a unit step ($u(t) = 0$ if $t < 0$, $u(t) = 1$ if $t \geq 0$), and $x(0) = [0,0]^T$, find $x(t)$. [40%]

2 A personal transporter (like a Segway) makes an angle θ with the vertical, and moves in a straight line with velocity v. The transporter is driven by a torque u applied to the wheels. For small angles the linearised equations of motion are

$$\begin{aligned} \ddot{\theta} &= \theta + v + u \\ \dot{v} &= \theta - v - u \end{aligned}$$

(a) Define a suitable state vector, keeping the state dimension as small as possible, and write these equations in the standard state-space form: $\dot{x} = Ax + Bu$. [15%]

(b) Show that the state-space system is controllable from the torque u. [15%]

(c) Feedback of the form

$$u = -k_1\theta - k_2\dot{\theta} - k_3 v$$

is applied. Verify that the feedback gains

$$k_1 = 12, \qquad k_2 = 9, \qquad k_3 = 4$$

give closed-loop asymptotic stability, and that the resulting closed-loop poles are located at $-1, -2, -3$. [40%]

(d) In order that the transporter should hold a desired speed even when ascending a hill, it is proposed that integral action should be used. Explain, with reference to the transporter, how integral action can be introduced into the state-feedback framework. (You are not expected to find any state feedback gains.) [30%]

3 The small-signal linearised model of an aircraft gives the following transfer function from the elevator angle δ to the pitch angle θ:

$$G(s) = \frac{(s+0.1)(s+0.2)}{(s^2+0.02)(s^2+s+5)}$$

It is proposed to improve the handling qualities by using constant-gain feedback, as shown in Fig.1. It is required that all the closed-loop poles have damping factors no smaller than $1/\sqrt{2}$. Figure 2 shows a sketch of the root-locus diagram for $G(s)$ (not drawn to scale).

(a) Explain why this proposal will *not* result in an acceptable closed-loop system. (*Hint:* Consider the damping factors associated with the open-loop poles.) [20%]

(b) An alternative proposal is to use feedback from the pitch rate $q = \dot{\theta}$ rather than the pitch angle θ, as shown in Fig.3.

(i) Find the transfer function $H(s)$ from the elevator angle δ to the pitch rate q. [10%]

(ii) Sketch the root-locus diagram for $H(s)$. (It is not necessary to locate any breakaway points accurately.) [30%]

(iii) Explain why this proposal will result in acceptable closed-loop damping, for a suitably chosen value of the feedback gain. [15%]

(c) Describe what happens if the pilot command is held at some constant value, after a period of being manipulated, with each of the feedback schemes shown in Figs. 1 and 3. [25%]

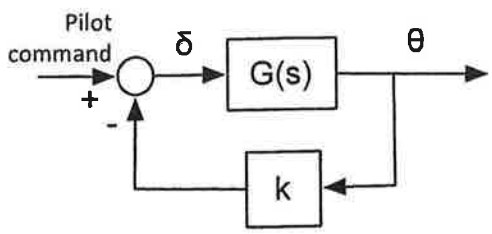

Fig. 1

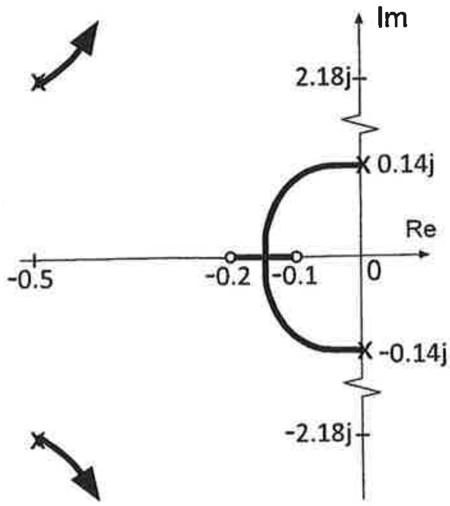

Fig. 2

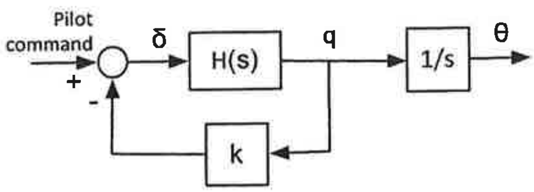

Fig. 3

4 Consider a linear system in standard state-space form:

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du$$

(a) Define what it means for such a system to be *observable*, and state how its observability can be checked. [20%]

(b) Show that if the standard rank test for observability is satisfied, then the system is observable. [30%] 40%?

(c) A linearised model of a small aircraft in a particular cruising condition is:

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -5 & 0 & -2 & 0 \\ -100 & 100 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -0.5 \\ 0 \\ -20 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -100 & 100 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The measured outputs are the pitch angle, the altitude, and the altitude rate (rate of climb), respectively. Show that this model is observable. [30%] 20%?

(d) Suppose that the altitude rate sensor fails. Determine whether the model in part (c) is observable from the remaining measurements, namely the pitch angle and altitude. [20%]

END OF PAPER

Version: final

9·12 = 9/120 = 21.6

3F2 Systems and Control: 2012 Answers

Prof. J.M. Maciejowski

7 June 2012

1. (a) —
   (b) —
   (c) —
   (d) —
   (e)
   $$x(t) = \begin{bmatrix} 1 - e^{-t} \\ 1 - 2e^{-t} + e^{-2t} \end{bmatrix}$$

2. (a)
   $$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} u$$
   (b) —
   (c) —
   (d) —

3. (a) —
   (b) i.
   $$H(s) = \frac{s(s+0.1)(s+0.2)}{(s^2+0.02)(s^2+s+5)}$$
   ii. —
   iii. —
   (c) —

4. (a) —
   (b) —
   (c) —
   (d) Observable.