3A1

2013

CRIB

18 pages

WND

24/1/2013.

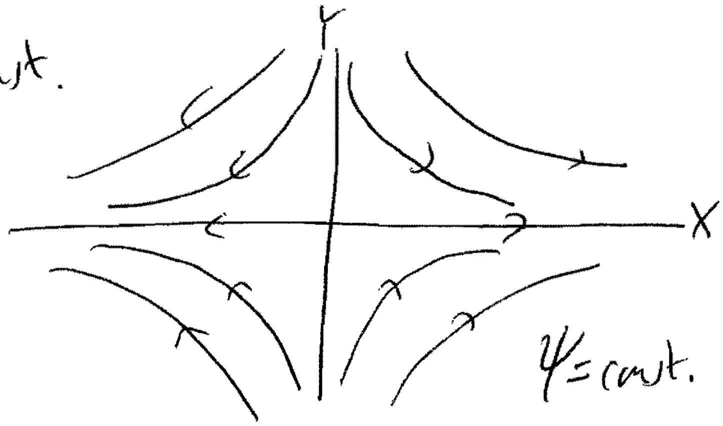
14/3/2013.

1 (a)(i) $u = \frac{\partial \psi}{\partial y} = Ax \rightarrow \psi = Axy + f(y)$

$v = -\frac{\partial \psi}{\partial x} = -Ay \rightarrow \psi = Axy + g(x)$

$\left.\right\}$ $\underline{\psi = Axy}$.

(ii) $xy = \text{const.}$



(iii) for inviscid flow the wall b/c is zero normal velocity so any streamline could be a wall — for our purposes, $y = 0$.

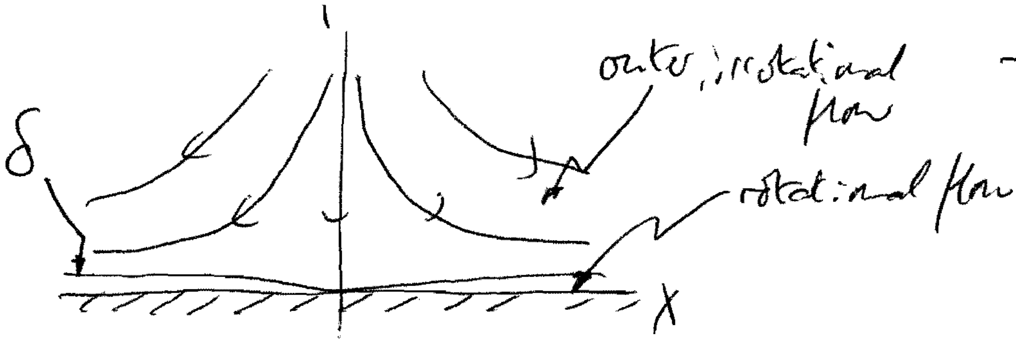
(b)(i) y-component of N-S: $\frac{1}{\rho}\frac{\partial p}{\partial y} = \nu \nabla^2 v - u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y}$

If the streamfunction, $\psi = x f(y)$ then

$$u = \frac{\partial \psi}{\partial y} = x f' \text{ and } v = -\frac{\partial \psi}{\partial x} = -f$$

$$\therefore \frac{1}{\rho}\frac{\partial p}{\partial y} = \nu\Big(\underbrace{\cancel{\frac{\partial^2 f}{\partial x^2}}}_{0} + \frac{\partial^2 f}{\partial y^2}\Big) - x f' \underbrace{\cancel{\frac{\partial f}{\partial x}}}_{0} - -f\frac{\partial -f}{\partial y}$$

$$\therefore \frac{1}{\rho}\frac{\partial p}{\partial y} = -\nu f'' - f f'$$ which is a function of y only and hence $\frac{\partial p}{\partial x}$ is a function of x only and therefore takes the outer flow form.



Hence: $p = p_0 - \frac{1}{2}\rho\left((Ax)^2 + (-A\cancel{Y})^2\right)$

$Y=0$ on wall

$$\therefore \frac{1}{\rho}\frac{\partial p}{\partial x} = -A^2 x$$

(ii) Turning to the X-component of N-S:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial Y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial Y^2}\right)$$

$$\therefore xf'^2 - xff'' = A^2 x + \nu x f'''$$

$$\text{i.e. } f'^2 - ff'' - \nu f''' = A^2$$

(c) $u(Y=\delta) = xf'(\delta)$ and $u(Y\to\infty) = Ax$ (outer flow)

So δ is defined by $\frac{xf'(\delta)}{Ax} = 0.99$

$$\therefore f'(\delta) = 0.99A$$

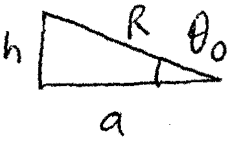
(which could be solved for δ)

Hence, δ is independent of x and the rotational region has constant thickness !!!

2. (a) Kutta condition: flow leaves the trailing edge smoothly, rather than passing round it, ie.

A range of irrotational flows about a flat plate can be derived by conformal mapping of irrotational flows of varying circulation about a cylinder. The Kutta condition allows us to select a physically meaningful value for the circulation.

(b) (i)



$$\left.\begin{aligned} a &= R\cos\theta_0 \\ h &= R\sin\theta_0 \end{aligned}\right\} \; a - ih = R(\cos\theta_0 - i\sin\theta_0) = Re^{-i\theta_0} \quad \text{QED.}$$

(ii) $$\frac{df}{dz} = U\left[e^{-i\alpha} - \frac{\cancel{U}R^2}{(z-ih)^2}e^{+i\alpha}\right] - \frac{i\Gamma}{2\pi}\cdot\frac{1}{z-ih}$$

So as $|z| \to \infty$, $\frac{df}{dz} = u - iv \to Ue^{-i\alpha}$, ie a flow with speed U at angle α to the real axis; OK ✓.

And, Γ is the circulation.

On the cylinder $z = ih + Re^{i\theta}$ where:

$$\left.\begin{array}{l} e^{i\theta} = c + is \\ e^{-i\theta} = c - is \end{array}\right\}$$

$$F = Uihe^{-i\alpha} + URe^{i(\theta-\alpha)} + \frac{UR^2}{Re^{i\theta}} e^{i\alpha} - i\frac{\Gamma}{2\pi}\ln(Re^{i\theta})$$

$$= Uih\underbrace{e^{-i\alpha}}_{\cos\alpha - i\sin\alpha} + UR\underbrace{[e^{i(\theta-\alpha)} + e^{-i(\theta-\alpha)}]}_{2\cos(\theta-\alpha)} - \frac{i\Gamma}{2\pi}\ln R + \frac{\Gamma\theta}{2\pi}$$

$F = \phi + i\psi$, so on the cylinder, $\psi = \mathrm{Im}(F)$

$\therefore \psi_{cyl} = Uh\cancel{\sin}\cos\alpha - \frac{\Gamma}{2\pi}\ln R$ = constant : OK ✓.

(iii) Conformal mapping allows us to deduce the complex potential in the ζ-plane:

$$\frac{dF}{d\zeta} = \frac{dF}{dz}\cdot\frac{dz}{d\zeta} = \underbrace{\frac{dF}{dz}}_{\text{see (b)ii}} \frac{1}{1 - a^2/z^2} \rightarrow \infty \text{ as } z \rightarrow \pm a$$

Kutta requires $F'(z=a) = 0$, i.e.

$$\frac{i\Gamma}{2\pi} / \underbrace{(a-ih)}_{\substack{Re^{i\theta_0} \\ \text{see (b)i}}} = U\left[e^{-i\alpha} - \frac{R^2}{(a-ih)^2} e^{+i\alpha}\right]$$

$$\therefore \frac{i\Gamma}{2\pi} = UR\left[e^{-i(\alpha+\theta_0)} - e^{+i(\alpha+\theta_0)}\right] = -2iUR\sin(\alpha+\theta_0)$$

$\therefore \Gamma = -4\pi UR\sin(\alpha+\theta_0)$ and Lift $= -\rho U\Gamma$

$$= 4\pi\rho U^2 R\sin(\alpha+\theta_0)$$

(c) At the leading edge ($z=-a$) $\left|\frac{dz}{d\zeta}\right| \to \infty$ so infinite velocities are predicted. In reality the associated extreme adverse pressure gradient would result in flow separation. In our model this is avoidable if $F'(z=-a)=0$ also.

$$\text{i.e. } -2i\pi R\sin(\alpha+\theta_0) = \pi\left[\frac{R^2}{a+ih}e^{i\alpha} - (a+ih)e^{-i\alpha}\right]$$

$$= +2i\pi R\sin(\alpha-\theta_0)$$

$$\therefore \quad \underline{\alpha = 0}.$$

3. (a) Kelvin: in an inviscid flow, with uniform density, the circulation around a closed loop which moves with the fluid is constant

i.e. $\frac{D\Gamma}{Dt} = 0$.

If the flow is initially irrotational, Stokes' theorem tells us that the circulation around any closed loop is zero; Kelvin then implies that it remains zero.

(b) (i) Within the bend $\bar{u} = u_r(r)\hat{e}_r + u_\theta(r)\hat{e}_\theta$ independent of the azimuthal angle θ (given).

Continuity: $\nabla\cdot\bar{u} = \frac{1}{r}\frac{d}{dr}(ru_r) + \underbrace{\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}}_{0} = 0$

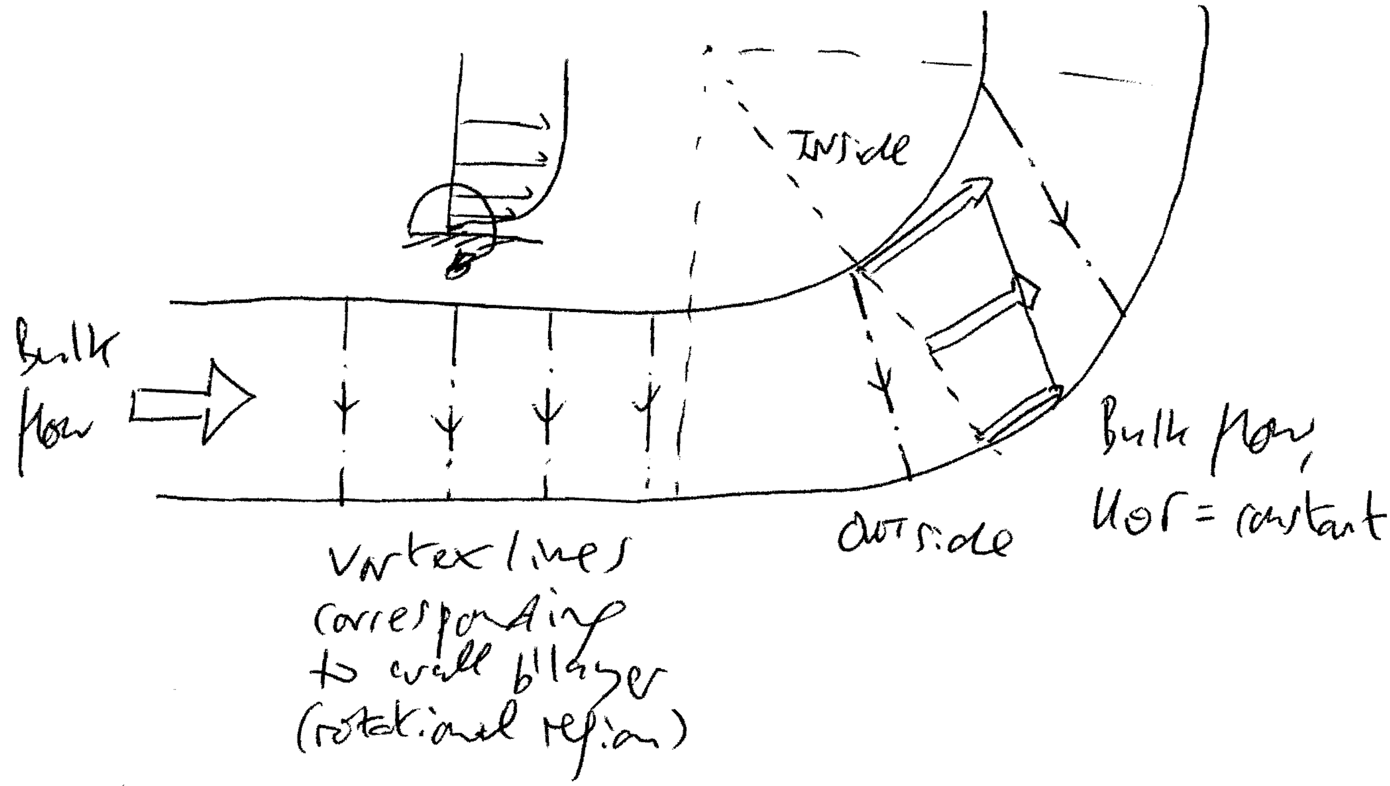
$\therefore$ u_r = constant across bend. Since $u_r = 0$ on the inner and outer walls, $u_r = 0$ for all r. Hence the streamlines are circular.

(ii) The bulk flow is irrotational, $\nabla\times\bar{u} = 0$

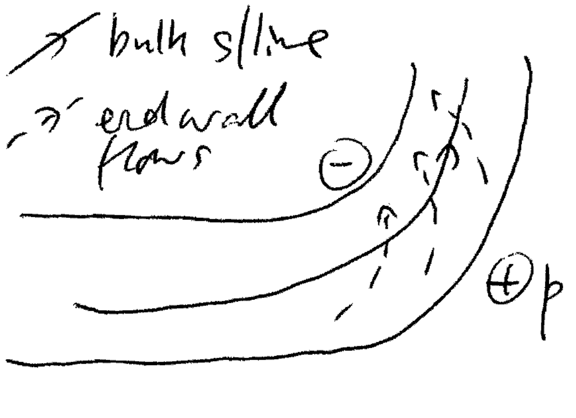
$\therefore$ $\frac{1}{r}\left(\frac{d}{dr}(ru_\theta) - \underbrace{\frac{\partial u_r}{\partial\theta}}_{0\text{ (given)}}\right) = 0$

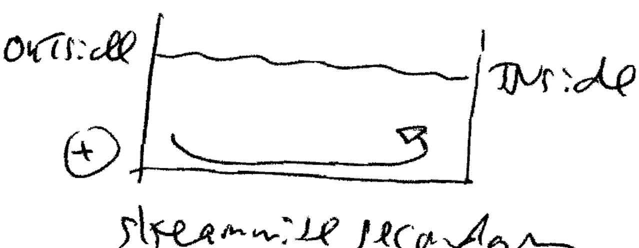
$\therefore$ $\underline{ru_\theta = \text{constant}}$ (as expected; free vortex).

(c)



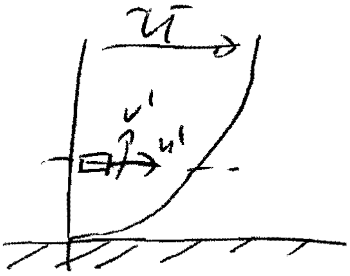
The bulk outer flow is slower than the inner ($U\theta r$=const) hence vortex lines become increasingly rotated relative to radial lines ($D\Gamma/Dt=0$). Thus a streamwise component of vorticity – secondary flow – develops within the endwall b'layer





streamwise secondary flow in endwall b'layer.

4. (a) Consider an element of fluid within the b'layer. The Reynolds' stress, $-\overline{u'v'}$, is negative in sign, but the stress, τ, positive; hence:



(1) $u'>0$, $v'<0$ so the fluid element moves faster than local mean, $-$ve v' pulls this higher momentum fluid towards wall

(2) $u'<0$, $v>0$ —"— vice versa ...

$\Rightarrow$ both cases mean that higher momentum fluid is entrained towards the wall so turbulent b'layers have steeper gradients and higher shear stresses.

(b)(i) Near the wall, $Y \to 0$ and for zero pressure gradient the b'layer equation:

$$\bar{u}\frac{\partial \bar{u}}{\partial X} + \bar{v}\frac{\partial \bar{u}}{\partial X} = -\frac{\partial}{\partial Y}(\tau/\rho)$$

At wall: $\bar{u}=0$ & $\bar{v}=0$ $\therefore \frac{d}{dY}\left(\frac{\tau_w}{\rho}\right)=0 \Rightarrow \frac{\tau_w}{\rho}=\text{const.}$

and: $u'=0$, $v'=0$ $\therefore -\rho\overline{u'v'}=0$ hence $\tau = \mu\frac{d\bar{u}}{dY}$

Hence: $\frac{\mu}{\rho}\frac{d\bar{u}}{dY} = \frac{\tau_w}{\rho} \Rightarrow \bar{u} = \frac{(\tau_w/\rho)\,Y}{\nu}$ u_τ^2

$\therefore$ $\underline{\bar{u}/u_\tau \sim u_\tau Y/\nu}$; viscous sublayer.

(ii) Further away from the wall the velocity profile is dominated by $-\overline{u'v'}$. The law of the wall is:

$$\frac{\bar{u}}{u_\tau} = f\left(\frac{u_\tau y}{\nu}\right)$$

$$\therefore \frac{d\bar{u}}{dy} = u_\tau f'\left(\frac{u_\tau y}{\nu}\right)\frac{u_\tau}{\nu} = \frac{u_\tau^2}{\nu} f'\left(\frac{u_\tau y}{\nu}\right)$$

Given: $A = \frac{y}{u_\tau}\frac{d\bar{u}}{dy} \Rightarrow \frac{d\bar{u}}{dy} = \frac{A u_\tau}{y} = \frac{u_\tau^2}{\nu} f'(\;)$

$$\therefore A \cdot \underbrace{\frac{\nu}{u_\tau y}}_{} = f'\Big(\underbrace{\frac{u_\tau y}{\nu}}_{y^+,\ \text{the wall coordinate.}}\Big)$$

$$\therefore \int \frac{A}{y^+} dy = \int \frac{df}{dy} \cdot dy \quad \left(\text{with } \frac{dy}{dy^+} \text{ cancelled} \ldots\right)$$

Whence: $f\left(\frac{u_\tau y}{\nu}\right) = \underline{\frac{\bar{u}}{u_\tau} = A \log y^+ + C}$.

This is the so-called log law.

5. (a) The local lift coefficient is related to the local effective angle of attack by

$$C_l(y) = 2\pi\alpha_{eff} = 2\pi\,(\underbrace{\alpha}_{\text{AoA}} - \underbrace{\alpha_0(y)}_{\text{TWIST}} - \underbrace{\cancel{\alpha_d}(y)}_{\text{DOWNWASH ANGLE}})$$

+ local lift, $l = \rho U \Gamma(y)$

Whence the "lifting line equation" ($C_l = \frac{l}{\frac{1}{2}\rho U^2 c}$)

$$\frac{\Gamma(y)}{\pi U c(y)} = \alpha - \alpha_0(y) - \alpha_d(y)$$

(b) Elliptic loading $\Gamma(y) = \Gamma_0\left(1-\frac{y^2}{s^2}\right)^{1/2} \rightarrow \Gamma(\phi) = \Gamma_0 \sin\phi$

for $\eta = -s\cos\phi$.

& $y = -s\cos\theta$

Downwash, $w_d(y) = \frac{1}{4\pi}\int_{-s}^{+s} \frac{d\Gamma}{d\eta}\cdot\frac{d\eta}{y-\eta}$

$$= \frac{\Gamma_0}{4\pi s}\int_0^\pi \frac{\cos\phi\, d\phi}{\cos\phi - \cos\theta}$$

π; Glauert (DATA)

$\therefore\ \alpha_d = \frac{w_d}{U} = \frac{\Gamma_0}{4sU}$, constant across span.

Hence

$$\underbrace{\frac{\Gamma_0\sqrt{1-y^2/s^2}}{\pi U\, c(y)}}_{} = \underbrace{\alpha}_{} - \alpha_0(y) - \underbrace{\frac{\Gamma_0}{4Us}}_{\text{constant}}$$

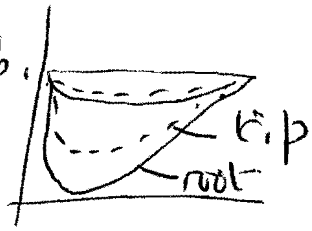
So (i) at constant twist, chord varies elliptically (eg. $c(y) = c_0\sqrt{1-y^2/s^2}$ "SPITFIRE")

or (ii) at constant chord, twist varies elliptically.

$\therefore$ (iii) if <u>both</u> chord <u>and</u> twist are constant

Then local $C_\ell = \dfrac{\rho U \Gamma(y)}{\frac{1}{2}\rho U^2 c} = \dfrac{2\Gamma_0}{Uc}\sqrt{1-y^2/s^2}$

i.e. the spanwise loading variation must be derived from varying section design — as sketched



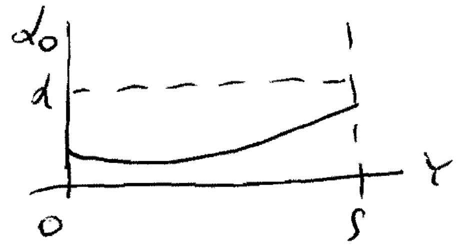
(c) $\underbrace{\dfrac{\Gamma_0\sqrt{1-s^2/y^2}}{\pi U c_0(1-\frac{y}{2s})}}_{\text{linear taper}} + \dfrac{\Gamma_0}{4Us} = \alpha - \alpha_0(y)$;

$y=0 \rightarrow \alpha_0 = \alpha - \dfrac{\Gamma_0}{U}\left(\dfrac{1}{4s} + \dfrac{1}{\pi c_0}\right)$

$y=\dfrac{s}{2} \rightarrow \alpha_0 = \alpha - \dfrac{\Gamma_0}{U}\left(\dfrac{1}{4s} + \dfrac{1}{\pi c_0}\cdot\dfrac{2}{\sqrt{3}}\right)$

$y=s \rightarrow \alpha_0 = \alpha - \dfrac{\Gamma_0}{U}\left(\dfrac{1}{4s}\right)$ — sketched

Hence:



(d) Local $C_l = 2\pi(\alpha - \underbrace{\alpha_0 - \frac{\Gamma_0}{4Us}}_{\text{constant}})$

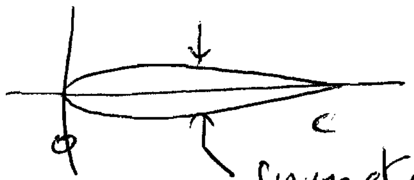
$\therefore$ global $C_L = 2\pi\left(\alpha-\alpha_0-\dfrac{\Gamma_0}{4Us}\right)\cancel{2s}$

Now, total lift, $L = \displaystyle\int_{-s}^{+s}\rho U\Gamma\,dy = \rho U\Gamma_0\int_0^{\pi}\sin\theta\,\underbrace{s\sin\theta\,d\theta}_{dy} = \rho U\Gamma_0\frac{\pi s}{2}$

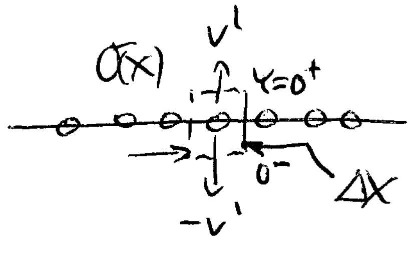
$\therefore C_L = \dfrac{L}{\frac{1}{2}\rho U^2 cs} = \Gamma_0\dfrac{\pi}{Uc}$

$\therefore C_L = 2\pi(\alpha-\alpha_0) - \overbrace{\dfrac{\cancel{2\pi}}{2\cancel{4}\cancel{U}s}\cdot\dfrac{\cancel{U}c}{\cancel{\pi}}}^{1/R}C_L$

$\therefore C_L = \left(\dfrac{1}{1+\frac{1}{R}}\right)2\pi(\alpha-\alpha_0)$; as $R\to\infty$, 2D result is recovered.

6. (a) 

Streamlines: $\left.\dfrac{v'}{U}\right|_{y=0^+} = \dfrac{1}{2}\dfrac{dt}{dx}$; $\left.\dfrac{v'}{U}\right|_{y=0^-} = -\dfrac{1}{2}\dfrac{dt}{dx}$.

(b) 

source per unit chord is $\sigma(x)$
So volume flow: $\sigma \Delta x = 2v' \Delta x$

$\therefore \sigma(x) = U\dfrac{dt}{dx}$.

(c) The u' velocity at location $(x,0)$ is given by:
$$u'(x,0) = \int_0^c \frac{\sigma(x')}{2\pi(x-x')}\,dx'.$$

(d) Given the thickness distribution, $t = c\sum_{n=1}^{\infty} \tau_n \sin(n\theta)$

$$u' = \int_0^\pi \underbrace{\left(Uc\sum_n n\tau_n \frac{\cos(n\theta)}{-\frac{c}{2}\sin\theta}\right)}_{U dt/dx} \cdot \frac{\overbrace{-\frac{c}{2}\sin\theta\, d\theta}^{dx'}}{2\pi\frac{c}{2}(\cos\theta-\cos\phi)}$$

$x = \frac{c}{2}(1+\cos\theta) \Rightarrow$
$dx = -\frac{c}{2}\sin\theta\, d\theta$

$\left.\begin{matrix} x=0, \theta=\pi \\ x=c, \theta=0 \end{matrix}\right\}$

$$= \frac{U}{\pi}\sum_n n\tau_n \int_0^\pi \frac{\cos n\theta\, d\theta}{\cos\theta-\cos\phi}$$

Glauert (DATA) $\pi\dfrac{\sin(n\theta)}{\sin\theta}$

$$\therefore \frac{u'}{U} = \sum_{n=1}^{\infty} n\tau_n \frac{\sin(n\theta)}{\sin\theta}.$$

(e) ~ NACA 2409

τ_1	τ_2	τ_3	τ_4
+0.0780	−0.0216	−0.0017	−0.0013

(given)

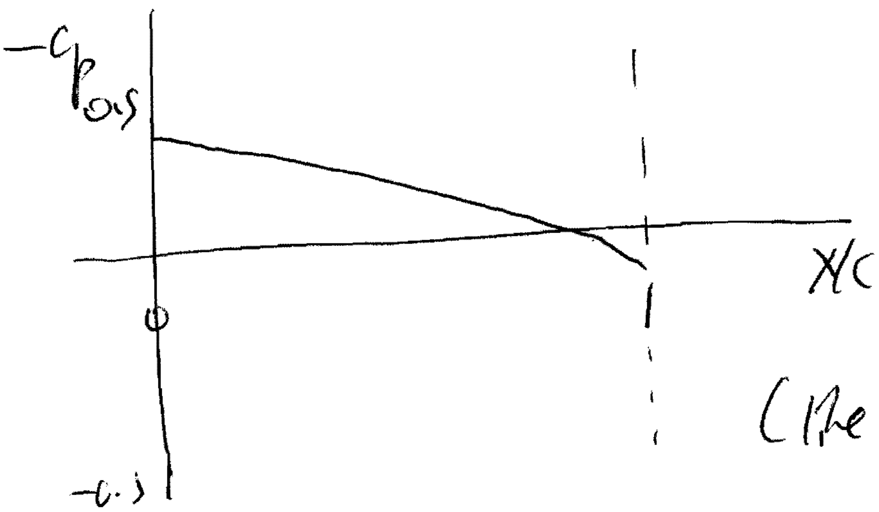
		LE ($\theta=\pi$)	TE ($\theta=0$)
$n=1$	$\frac{\sin\theta}{\sin\theta}=1$	1	1
$n=2$	$\frac{\sin 2\theta}{\sin\theta}=2\cos\theta$	2	−2
$n=3$	$\frac{\sin 3\theta}{\sin\theta}=2\cos^2\theta+\cos 2\theta$	3	3
$n=4$	$\frac{\sin 4\theta}{\sin\theta}=4\cos\theta(2\cos^2\theta-1)$	4	−4

"sidewarting"!!!

$$C_p = -2u'/\pi$$

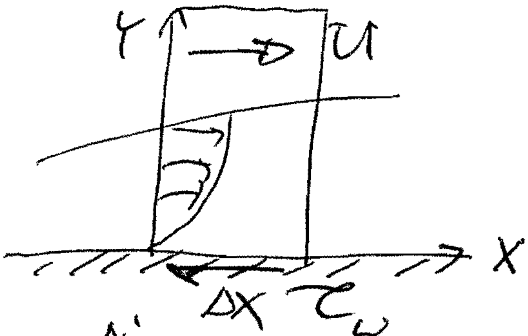
Hence, at LE, $\frac{u'}{\pi} = \begin{array}{l}0.0780\\+0.1024\\-0.0153\\-0.0208\end{array} = 0.1859 \Rightarrow C_{p_{LE}} = -0.37$

at TE, $\frac{u'}{\pi} = -0.039 \Rightarrow C_{p_{TE}} = +0.08$



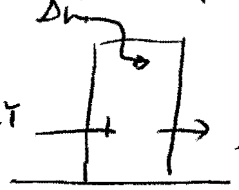
(the lecture notes, Fig. 20

7. (a)



Integral approach:

• Mass flow:



$\int \rho u\,dy \rightarrow \int \rho u\,dy + \frac{d}{dx}\left(\int \rho u\,dy\right)\delta x$ $\quad\therefore \delta\dot m = \frac{d}{dx}\left(\int \rho u\,dy\right)\delta x$

• Momentum:

$$\cancel{\int \rho u^2 dy} + \cancel{\int p\,dy} + \overbrace{U\frac{d}{dx}\left(\int \rho u\,dy\right)\delta x}^{\delta\dot m U}$$

$$= \tau_w \delta x + \cancel{\int} + \frac{d}{dx}\int \rho u^2 dy\,\delta x + \cancel{\int} + \frac{d}{dx}\int p\,dy\,\delta x$$

$$\therefore U\frac{d}{dx}\int \rho u\,dy - \frac{d}{dx}\int \rho u^2 dy - \int \frac{dp}{dx} dy = \tau_w$$

$U = f(x)$ $\qquad \frac{dp}{dx} = -\rho U \frac{dU}{dx}$ Bernoulli in free stream

$$\therefore \frac{d}{dx}\int U u\,dy - \frac{dU}{dx}\int u\,dy - \frac{d}{dx}\int \rho u^2 dy + \int U\frac{dU}{dx} dy = \tau_w/\rho$$

$$\frac{d}{dx}\int (Uu - u^2)\,dy + \frac{dU}{dx}\int (U-u)\,dy = \tau_w/\rho \qquad U \neq f(y)$$

Now $\delta^* = \int \left(1-\frac{u}{U}\right)dy$; $\theta = \int \frac{u}{U}\left(1-\frac{u}{U}\right)dy$; $H = \delta^*/\theta$

So: $\frac{d}{dx}(U^2\theta) + U\delta^*\frac{dU}{dx} = \tau_w/\rho$

$U^2\frac{d\theta}{dx} + \theta\, 2U\frac{dU}{dx} + U H \theta \frac{dU}{dx} = \tau_w/\rho$; $C_f = \tau_w / \tfrac{1}{2}\rho U^2$

$$\therefore \frac{d\theta}{dx} + \frac{dU}{dx}\cdot\frac{H+2}{U}\cdot\theta = \frac{C_f}{2}$$

(see DATA).

(b) If: $\frac{u}{U} = \eta(2-\eta)$ where $\eta = Y/\delta^*$

$$\theta = \int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dY = \int_0^1 (2\eta-\eta^2)(1-2\eta+\eta^2)\,\overbrace{\delta^* d\eta}^{dY}$$

$$= \delta^*\int_0^1 \left(2\eta - \overset{5}{\cancel{4}}\eta^2 + \overset{4}{\cancel{2}}\eta^3 - \eta^4\right)d\eta$$

$$= \delta^*\left[\eta^2 - \frac{5}{3}\eta^3 + \eta^4 - \frac{1}{5}\eta^5\right]_0^1 = \delta^*\left(1+1-\frac{5}{3}-\frac{1}{5}\right)$$

$$\frac{30-25-3}{15}$$

$$\therefore \underline{\theta/\delta^* = 2/15}.$$

$$\tau_w = \mu\frac{du}{dY}\bigg|_{\eta=0} = \mu\frac{d}{d\eta}\left[U(2\eta-\eta^2)\right]\underbrace{\frac{d\eta}{dY}}_{\frac{1}{\delta^*}}$$

$$= \frac{\mu U}{\delta^*}(2-2\eta)\big|_{\eta=0} = \frac{2\mu U}{\delta^*}$$

$$\therefore \underline{C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{4\cdot(\mu/\rho)}{U\delta^*}}.$$

(c) From (a) $\frac{d\theta}{dX} = \frac{C_f}{2}$ at zero pressure gradient

Hence, from (b) $\frac{2}{15}\frac{d\delta^*}{dX} = \frac{2\overbrace{(\mu/\rho)}^{\nu}}{U\delta^*}$

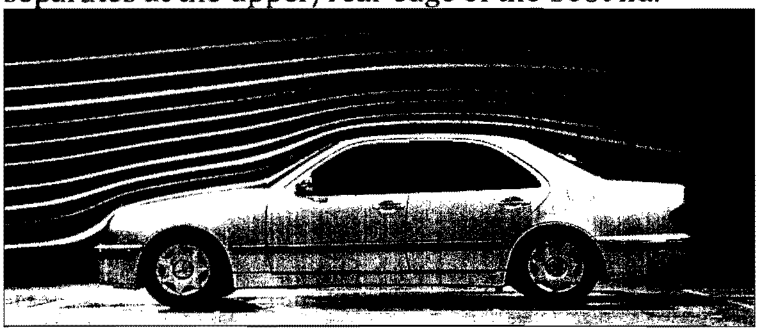
$$\therefore \int\delta^* d\delta^* = \int 15\frac{\nu}{U}dX$$

$$\therefore \underline{\delta^* \sim \sqrt{\frac{30\nu X}{U}}}.$$

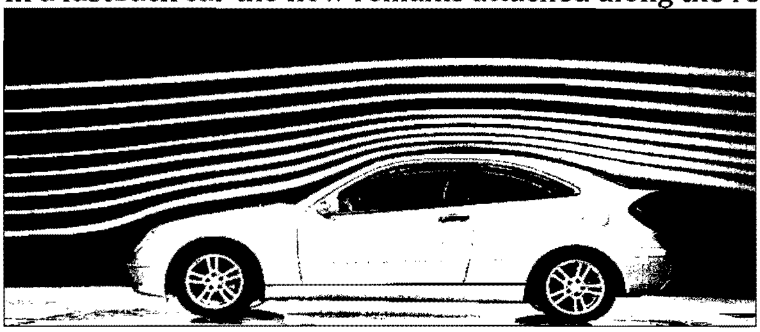
Solutions (The pictures are from the lecture handout).

8. a)
Hatchback (the picture is of an estate, but the rear flow is the same):
The flowfield is dominated by flow separation at the upper edge of the rear window and the whole base flow I sseparated:

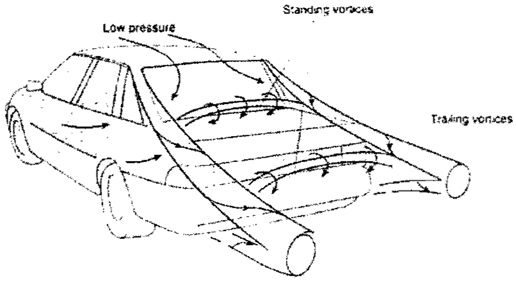
Saloon
In a well-designed saloon car, the flow stays attached along the rear window and separates at the upper/rear edge of the boot lid.

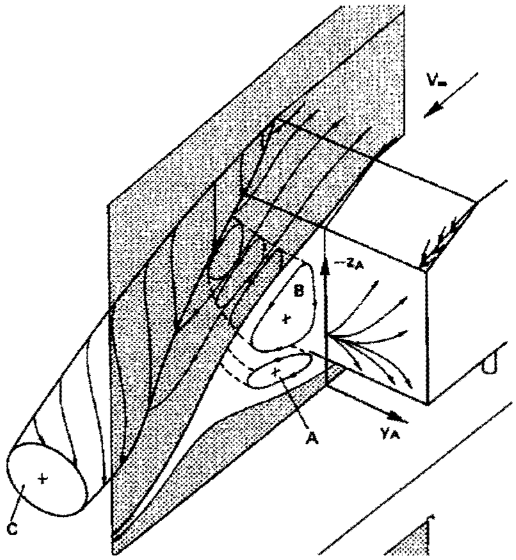


Fastback
In a fastback car the flow remains attached along the rear slope.



However, this typically leads to two large counter-rotating vortices (for a 'bonus point):



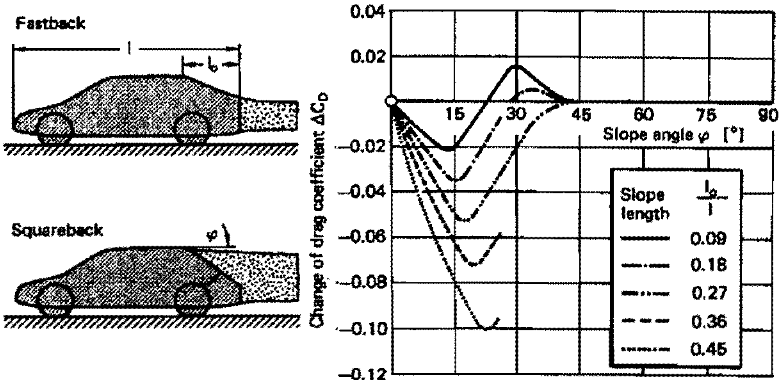


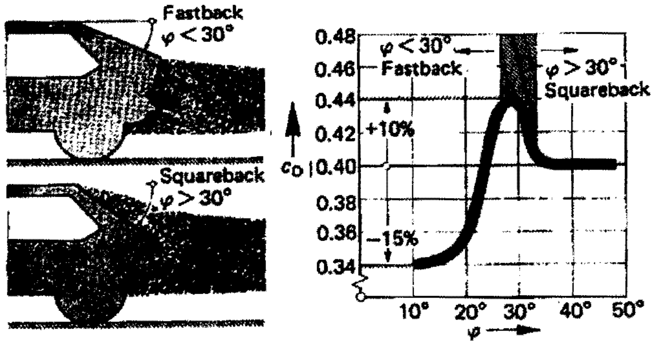
b)

- Hatchback: Large separation (at the roof line) but maximum internal volume (good for short cars). Not the best drag coefficient because of the large separation.
- Fastback: Attached flow along the rear window but possibly with strong vortices. Not good for volume (boot space), but good to excellent drag coefficient. Can have a lot of lift though, which may require spoilers in performance cars.
- Saloon: Attached flow along rear window and the boot lid. Separation at rear of boot lid. Can have vortices (but not as strong as fastback). Decent use of space through classical boot. If well designed this can give very low drag values.

c)

The angle of the rear window slope is crucial. At a slope of around 30degrees the flow changes from attached to separated:

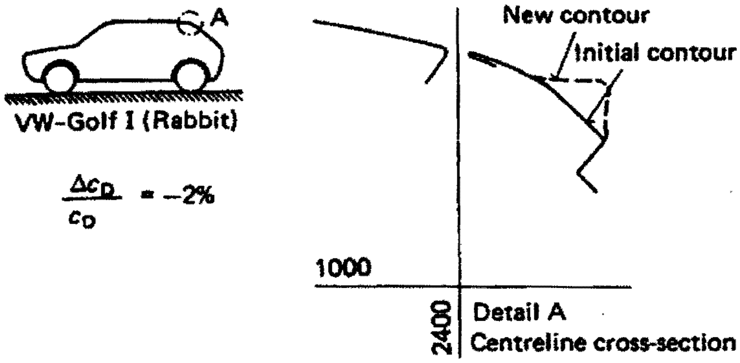




Typical angles for well-designed fastback shaped cars are around 20degrees.

d) There is usually a degree of boattailing to reduce the base area and recover some pressure. This means that the roofline drops slightly before the rear window and that the sides of the car are brought in.

In addition a distinct separation lip can help to slightly improve the base pressure (and thus reduce drag):



END.