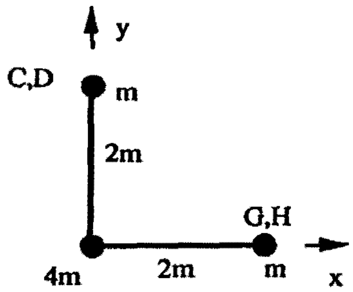
1.

From the data sheet $\quad \mathbf{I} = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}$

where $\quad A = \int (y^2+z^2)dm, \; B = \int (z^2+x^2)dm, \; C = \int (x^2+y^2)dm$

$D = \int yz\,dm, \; E = \int zx\,dm, \; F = \int xy\,dm$.



Viewing the shaft along the y--axis

$$\int z^2 dm = \int x^2 dm = ma^2 + 2 \times \tfrac{1}{3}ma^2 = \tfrac{5}{3}ma^2$$

$$\therefore B = \tfrac{10}{3}ma^2$$

x and z are never simultaneously non-zero, so $E = 0$.

For D, $\quad \int yz\,dm = m(-a)\tfrac{1}{2}a \text{ [DE] } + m(-\tfrac{3}{2}a)a \text{ [CD] } + m(-2a)\tfrac{1}{2}a \text{ [BC]}$

$$= ma^2(-\tfrac{1}{2} - \tfrac{3}{2} - 1) = -3ma^2$$

F likewise, but positive (this can be inferred from symmetry).

Thus $\quad \mathbf{I}_0 = ma^2 \begin{bmatrix} \frac{89}{3} & -3 & 0 \\ -3 & \frac{10}{3} & 3 \\ 0 & 3 & \frac{89}{3} \end{bmatrix}$

If $\mathbf{x}$ is a principal axis, $\mathbf{I}_0\mathbf{x} = \lambda\mathbf{x}$.

$$\mathbf{I}_0 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = ma^2 \begin{bmatrix} \frac{89}{3} \\ 0 \\ \frac{89}{3} \end{bmatrix} = \tfrac{89}{3}ma^2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

so $(1, 0, 1)$ is a principal axis, with moment of inertia $\frac{89}{3}ma^2$

b) By symmetry the centre of mass lies on $(1, 0, 1)$. For the x-component, take moments about y for x offsets.

$$10m\bar{x} = ma + 2 \times \tfrac{1}{2}ma = 2ma$$

$\bar{x} = \frac{a}{5}$ and $\bar{z} = \frac{a}{5}$ by symmetry, so the centre of mass is at $\frac{a}{5}(1,0,1)$.

By the parallel axes theorem, $\mathbf{I}_0 = \mathbf{I}_g + m\begin{bmatrix} y^2+z^2 & -yx & -zx \\ -xy & z^2+x^2 & -zy \\ -xz & -yz & x^2+y^2 \end{bmatrix}$

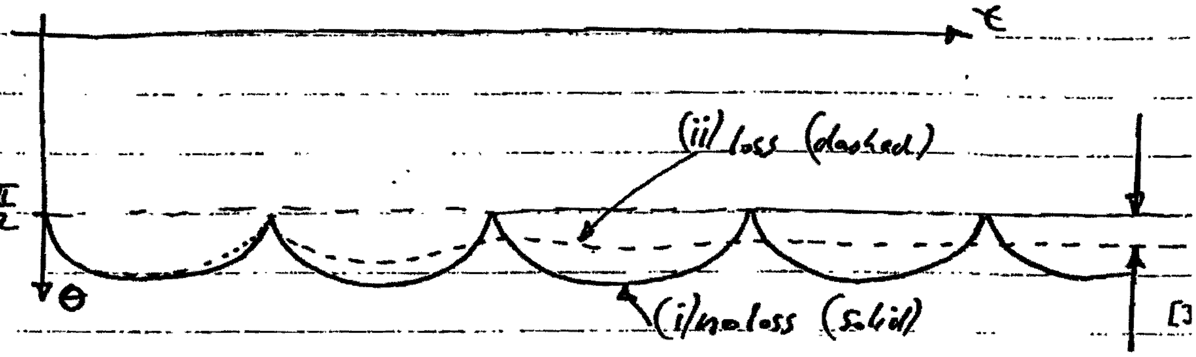
$$= \mathbf{I}_g + m(x^2+y^2+z^2)\mathbf{I} - m\mathbf{x}\mathbf{x}^T$$

$x^2+y^2+z^2 = 2a^2/25 \qquad$ and $\mathbf{x}\mathbf{x}^T = a^2/25 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} = a^2/25 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

so $\quad \mathbf{I}_g = \mathbf{I}_0 - \begin{bmatrix} \frac{ma^2}{25} & 0 & -\frac{ma^2}{25} \\ 0 & 2\frac{ma^2}{25} & 0 \\ -\frac{ma^2}{25} & 0 & \frac{ma^2}{25} \end{bmatrix} = ma^2 \begin{bmatrix} \frac{89}{3} - \frac{10}{25} & -3 & \frac{10}{25} \\ -3 & \frac{10}{3} - \frac{20}{25} & 3 \\ \frac{10}{25} & 3 & \frac{89}{3} - \frac{10}{25} \end{bmatrix} = ma^2 \begin{bmatrix} \frac{439}{15} & -3 & \frac{2}{5} \\ -3 & \frac{38}{15} & 3 \\ \frac{2}{5} & 3 & \frac{439}{15} \end{bmatrix}$

c) The principal moment of inertia about $(1, 0, 1)$ is $\frac{89}{3}$ ma^2 as found earlier, and applies about either O or about G.

2(a)



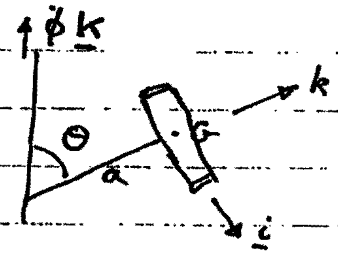
this is the permanent steady state value of the θ offset.

(b) To begin with, $KE = \frac{1}{2}C\omega^2$.

and $\underline{h}\cdot\underline{K} = 0$

ie there is no moment of momentum about a vertical axis.

Subsequent motion: PE + KE & $\underline{h}\cdot\underline{K}$ are conserved.



Velocity of G: $\underline{v} = a\dot\theta\,\underline{i} + a\sin\theta\,\dot\phi\,\underline{j}$

Angular velocity of rotor: $\underline{\omega} = \omega_1\underline{i} + \omega_2\underline{j} + \omega_3\underline{k}$

$\omega_1 = -\dot\phi\sin\theta$

$\omega_2 = \dot\theta$

$\omega_3 = \dot\phi\cos\theta + \omega \;=\; \omega$ for fast spin

$\therefore\; KE = \frac{1}{2}ma^2\left(\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right) + \frac{1}{2}A\left(\dot\phi^2\sin^2\theta\right) + \frac{1}{2}A\dot\theta^2 + \frac{1}{2}C\omega^2$

At extreme position $\dot\theta = 0$

$\therefore\; KE = \frac{1}{2}(ma^2 + A)\sin^2\theta\,\dot\phi^2 + \frac{1}{2}C\omega^2$

Put $\theta = \frac{\pi}{2} + \alpha \quad \therefore \quad \sin\theta = \cos\alpha \simeq 1$, $\quad \cos\theta = -\sin\alpha \simeq -\alpha$

Energy Conservation: $KE + PE = \frac{1}{2}(ma^2 + A)\dot\phi^2 + \frac{1}{2}C\omega^2 - mga\alpha = \frac{1}{2}C\omega$

$$\therefore\; \boxed{(ma^2 + A)\dot\phi^2 = 2mga\alpha} \qquad (1)$$

Moment of momentum: $\underline{h} = (A + ma^2)\omega_1\underline{i} + (A + ma^2)\omega_2\underline{j} + C\omega$

$\underline{h}\cdot\underline{K} = (A + ma^2)(-\dot\phi\sin\theta)(-\sin\theta) + C\omega\cos\theta = 0$

$\therefore\; (A + ma^2)\dot\phi\cos^2\alpha - C\omega\sin\alpha = 0$

(c) ... $\therefore \boxed{(A+ma^2)\dot{\phi} = C\omega\alpha}$ ②

eliminate $\dot{\phi}$ from ① & ②

$\frac{②^2}{①} \quad \therefore (A+ma^2) = \frac{C^2\omega^2\alpha}{2mga}$

$$\therefore \boxed{\alpha = \frac{2mga\,(A+ma^2)}{C^2\omega^2}}$$

(d) For steady motion (precession) small α

$$\therefore Q_2 = C\omega\dot{\phi}$$

$\therefore mga = C\omega\dot{\phi}$ ③

But ② (conservation of moment of momentum about $\underline{K}$) must always be satisfied

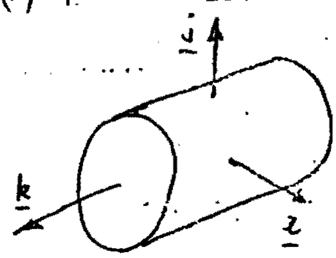
$\therefore \frac{②}{③}$ gives $\frac{A+ma^2}{C\omega} = \frac{C\omega\alpha_s}{mga}$

$$\therefore \alpha_s = \frac{mga\,(A+ma^2)}{C^2\omega^2}$$

Note that $\alpha = 2\alpha_s$

ie steady α is half the peak α reached.

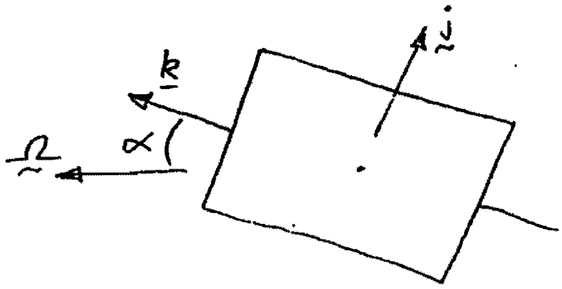
3/ (a)(i) From Mechanics data book page 19



$$A = B = \left(\frac{(d/2)^2}{4} + \frac{\ell^2}{12} \right) m$$

$$= \underline{\underline{\left(\frac{d^2}{16} + \frac{\ell^2}{12} \right) m}}$$

$$C = \frac{(d/2)^2}{2} m = \underline{\underline{\frac{d^2}{8} m}}$$ [3]



$$\underset{\sim}{\Omega} = \omega_1 \underline{i} + \omega_2 \underline{j} + \omega_3 \underline{k}$$

$$= -\Omega \sin\alpha \, \underset{\sim}{j} + \Omega \cos\alpha \, \underline{k}$$

$\therefore$ $\underline{\underline{\omega_1 = 0}}$ $\underline{\underline{\omega_2 = -\Omega \sin\alpha}}$ $\underline{\underline{\omega_3 = \Omega \cos\alpha}}$ [3]

(ii) Eulers equations from the data sheet

$$Q_1 = A\dot{\omega}_1 - (B - C)\,\omega_2 \omega_3$$

$$Q_2 = B\dot{\omega}_2 - (C - A)\,\omega_3 \omega_1$$

$$Q_3 = C\dot{\omega}_3 - (A - B)\,\omega_1 \omega_2$$

All these are zero for steady $\underset{\sim}{\Omega}$ and since $A = B$ and $\omega_1 = 0$ only the Q_1 equation is useful

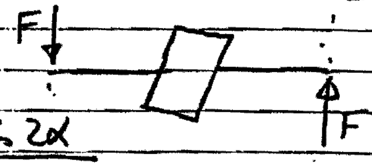
$$\therefore Q_1 = -(B - C)\,\omega_2 \omega_3$$

$$= -m\left(\frac{d^2}{16} + \frac{\ell^2}{12} - \frac{d^2}{8} \right)(-\Omega \sin\alpha)(\Omega \cos\alpha)$$

$$= m\Omega^2 \left(\frac{\ell^2}{12} - \frac{d^2}{16} \right) \sin\alpha \cos\alpha$$

$$= \frac{m\Omega^2 (4\ell^2 - 3d^2) \sin 2\alpha}{96}$$

This is a rotating couple about the i axis resulting in rotating bearing forces $\frac{Q_1}{L}$ at each bearing



$$F = \frac{M\Omega^2(4L^2-3d^2)\sin 2\alpha}{96L}$$

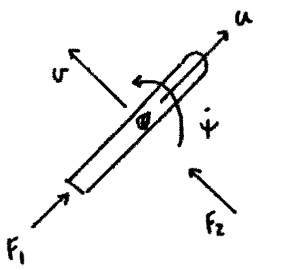
[12]

(iii) F is zero when $4L^2 = 3d^2$

$$\therefore \; L = \frac{\sqrt{3}}{2}d$$

In this case $A = B = C$ so the cylinder is dynamically equivalent to a sphere which generates no dynamic forces at any α. [2]

3 b)



For accelerations in u and v directions:

$$\ddot{\underset{\sim}{r}} = \ddot{\underset{\sim}{r}}_0 + \underset{\sim}{\omega} \times \dot{\underset{\sim}{r}}$$

↑ "non rotating" acceleration

$$\ddot{\underset{\sim}{r}} = (\dot u, \dot v, 0) + (0, 0, \dot\psi) \times (u, v, 0)$$

$$\ddot{\underset{\sim}{r}} = (\dot u - \dot\psi v, \dot v + \dot\psi u, 0)$$

ii Apply d'Alembert's principle and Force balance:

$$M(\dot u - \dot\psi v) = F_1$$
$$M(\dot v + \dot\psi u) = F_2$$

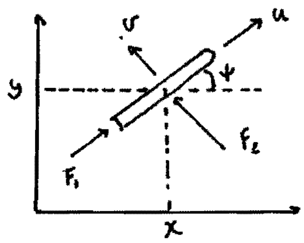
[20%]

Now for Lagrange, putting $\dot q_1 = u$ gives $q_1 = \int u\,dt$, but q_1 has no physical meaning since u is changing direction; the ship cannot be located by specifying q_1 and q_2.

Also: $T = \frac{1}{2}M(\dot q_1^2 + \dot q_2^2) \Rightarrow \frac{d}{dt}\left[\frac{\partial T}{\partial \dot q_1}\right] = M\ddot q_1 = M\dot u \Rightarrow$ "$\dot\psi v$" term missing from equation
$\Rightarrow$ Lagrange result incorrect

[10%]

(iii) Alternatively use cartesian displacements:



$$\dot x = u\cos\psi - v\sin\psi$$
$$\dot y = u\sin\psi + v\cos\psi$$

$$\ddot x = \dot u\cos\psi - \dot v\sin\psi - \dot\psi(u\sin\psi + v\cos\psi)$$
$$\ddot y = \dot u\sin\psi + \dot v\cos\psi + \dot\psi(u\cos\psi - v\sin\psi)$$

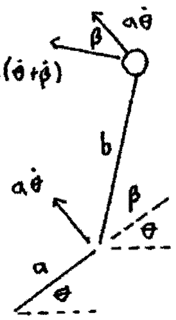
$T = \frac{1}{2}M(\dot x^2 + \dot y^2)$; $Q_x = F_1\cos\psi - F_2\sin\psi$
$Q_y = F_2\cos\psi + F_1\sin\psi$

Lagrange $\Rightarrow$ $M\ddot x = F_1\cos\psi - F_2\sin\psi$ — (1)
$M\ddot y = F_2\cos\psi + F_1\sin\psi$ — (2)

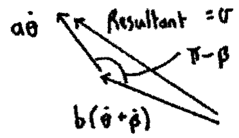
(1) × $\cos\psi$ + (2) × $\sin\psi$ $\Rightarrow$ $M(\dot u - \dot\psi v) = F_1$ ✓
−(1) × $\sin\psi$ + (2) × $\cos\psi$ $\Rightarrow$ $M(\dot v + \dot\psi u) = F_2$ ✓

[20%]

4 a)



Velocity diagram for Mass:



cosine rule $\quad v^2 = (a\dot\theta)^2 + b^2(\dot\theta+\dot\beta)^2 + 2ab\dot\theta(\dot\theta+\dot\beta)\cos\beta$

$\Rightarrow$ kinetic energy $\quad T = \frac{1}{2}M\left[a^2\dot\theta^2 + b^2(\dot\theta+\dot\beta)^2 + 2ab\dot\theta(\dot\theta+\dot\beta)\cos\beta\right]$

Potential energy $\quad V = Mg\left[a\sin\theta + b\sin(\theta+\beta)\right]$ [10%]

<u>Lagrange for θ</u>

$$\frac{d}{dt}\left[\frac{\partial T}{\partial\dot\theta}\right] - \frac{\partial T}{\partial\theta} + \frac{\partial V}{\partial\theta} = 0$$

$$\frac{d}{dt}\left[Ma^2\dot\theta + Mb^2(\dot\theta+\dot\beta) + 2Mab\dot\theta\cos\beta + Mab\dot\beta\cos\beta\right] + Mg\left[a\cos\theta + b\cos(\theta+\beta)\right] = 0$$

$$\underline{M(a^2+b^2)\ddot\theta + Mb^2\ddot\beta + Mab(2\ddot\theta+\ddot\beta)\cos\beta - Mab\dot\beta(2\dot\theta+\dot\beta)\sin\beta + Mga\cos\theta + Mgb\cos(\theta+\beta) = 0}$$

[15%]

<u>Lagrange for β</u>

$$\frac{d}{dt}\left[\frac{\partial T}{\partial\dot\beta}\right] - \frac{\partial T}{\partial\beta} + \frac{\partial V}{\partial\beta} = 0$$

$$\frac{d}{dt}\left[Mb^2(\dot\theta+\dot\beta) + Mab\dot\theta\cos\beta\right] + Mab\dot\theta(\dot\theta+\dot\beta)\sin\beta + Mgb\cos(\theta+\beta) = 0$$

$$Mb^2(\ddot\theta+\ddot\beta) + Mab\ddot\theta\cos\beta - Mab\dot\theta\dot\beta\sin\beta + Mab\dot\theta(\dot\theta+\dot\beta)\sin\beta + Mgb\cos(\theta+\beta) = 0$$

$$\underline{Mb^2(\ddot\theta+\ddot\beta) + Mab\ddot\theta\cos\beta + Mab\dot\theta^2\sin\beta + Mgb\cos(\theta+\beta) = 0}$$ [15%]

b) For $\dot\theta = \Omega$, $\ddot\theta = 0$, consider β equation:

$$Mb^2\ddot\beta + Mab\Omega^2\beta + \underbrace{Mgb\cos(\theta+\beta)} = 0$$

From $\sin\beta \approx \beta$ $\qquad$ Neglect since $Mab\Omega^2 \gg Mgb$ $\quad (\Omega^2 \gg g/a)$

$$\Rightarrow \quad Mb^2\ddot\beta + Mab\Omega^2\beta = 0 \quad \Rightarrow \quad \ddot\beta + \left(\frac{a}{b}\right)\Omega^2\beta = 0 \quad \Rightarrow \quad \underline{\omega_n^2 = \left(\frac{a}{b}\right)\Omega^2}$$ [20%]

4 cont c) Put $\theta = -\pi/2 + \hat{\theta}$, $\hat{\theta}$ = small vibration. $\cos\theta \approx \hat{\theta}$, $\cos(\theta+\beta) \approx \hat{\theta}+\beta$

Lagrange for θ, neglecting nonlinear terms:

$$M(a^2+b^2)\ddot{\theta} + Mb^2\ddot{\beta} + Mab(2\ddot{\theta}+\ddot{\beta}) + Mga\hat{\theta} + Mgb(\hat{\theta}+\beta) = 0$$

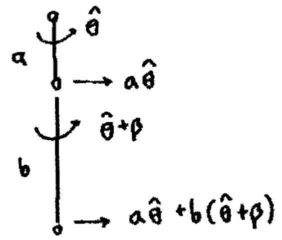
Lagrange for β, neglecting nonlinear terms:

$$Mb^2\ddot{\theta} + Mb^2\ddot{\beta} + Mab\ddot{\theta} + Mgb(\hat{\theta}+\beta) = 0$$

In Matrix form:

$$\begin{pmatrix} M(a+b)^2 & Mb(a+b) \\ Mb(a+b) & Mb^2 \end{pmatrix}\begin{pmatrix} \ddot{\hat{\theta}} \\ \ddot{\beta} \end{pmatrix} + \begin{pmatrix} Mg(a+b) & Mgb \\ Mgb & Mgb \end{pmatrix}\begin{pmatrix} \hat{\theta} \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ [20%]

Now when $b\beta = -(a+b)\hat{\theta}$ the mass does not move horizontally:



$a\hat{\theta} + b(\hat{\theta}+\beta) = 0$ when $b\beta = -(a+b)\hat{\theta}$

$\beta = -\left(\frac{a+b}{b}\right)\hat{\theta}$ has <u>no kinetic energy</u> and $[M]\begin{pmatrix} \ddot{\hat{\theta}} \\ -\left(\frac{a+b}{b}\right)\ddot{\hat{\theta}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{\omega_n = \infty}$ [20%]