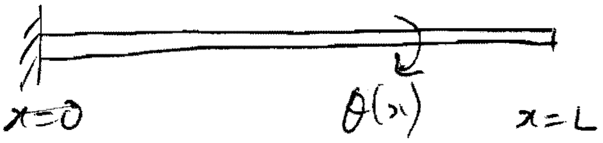
Engineering Tripos 2013 Part IIA 3C6 solutions

1(a)



Governing equation is $G \dfrac{\partial^2 \theta}{\partial x^2} = \rho \dfrac{\partial^2 \theta}{\partial t^2}$

For a mode, let $\theta(x,t) = u(x)\, e^{i\omega t}$

Then $u'' = -\dfrac{\omega^2}{c^2} u$ where $c^2 = \dfrac{G}{\rho}$

$\therefore u = A \sin \dfrac{\omega x}{c} + B \cos \dfrac{\omega x}{c}$, A and B constants.

At $x=0$, $u=0$ $\therefore B=0$

At $x=L$, $\dfrac{\partial u}{\partial x} = 0$ $\therefore A \dfrac{\omega}{c} \cos \dfrac{\omega L}{c} = 0$

$\therefore \dfrac{\omega L}{c} = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2} \ldots$

Mode shapes (sketched as transverse for clarity):

(35%)

(b) Torque at $x=0$ is $GI \left.\dfrac{\partial \theta}{\partial x}\right|_{x=0}$

where I is the polar moment of area.

From mechanics data book, $I = \dfrac{\pi}{2}\left(a^4 - (a-h)^4\right)$

Let $K = GI = \dfrac{G\pi}{2}\left(a^4 - (a-h)^4\right)$.

Then boundary condition is $K \dfrac{\partial \theta}{\partial x} = J \dfrac{\partial^2 \theta}{\partial t^2}$ at $x=0$

$\therefore$ for a mode, $Ku' = -J\omega^2 u$

Again, general solution is $u = A \sin \dfrac{\omega x}{c} + B \cos \dfrac{\omega x}{c}$

Substituting, $\quad KA\dfrac{\omega}{c} = -\omega^2 J B \quad (1)$

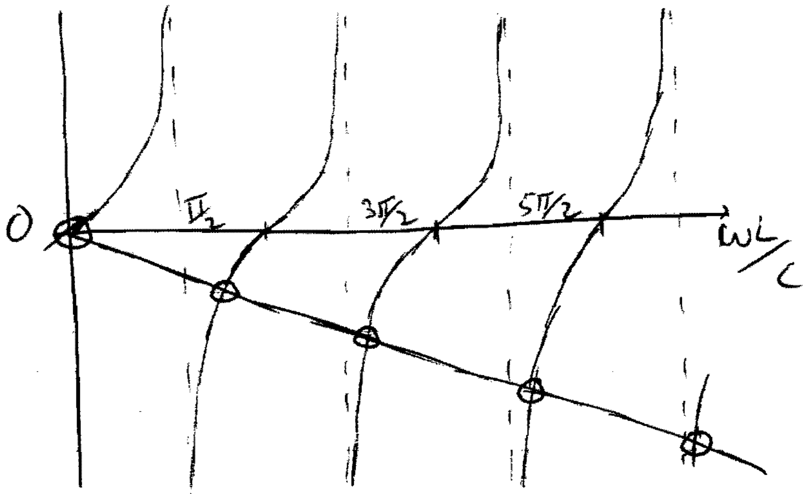
At $x=L$, $\quad A\dfrac{\omega}{c} \cos \dfrac{\omega L}{c} - B\dfrac{\omega}{c} \sin \dfrac{\omega L}{c} = 0$

$\therefore \dfrac{A}{B} = \tan \dfrac{\omega L}{c}$

Substitute in ①: $\tan \frac{\omega L}{c} = -\frac{Jc}{K}\omega$

The roots of this give the new natural frequencies. (35%)

(c) Plot $\tan \frac{\omega L}{c}$ and $-\frac{Jc}{K}\omega$ as functions of ω and see where they cross:



There is a rigid body mode at O, then all the other modes are a little higher in frequency than those of part (a), which occur at the dashed lines.

This means the wavelength is slightly shorter than in part (a)

So the first three non-zero frequencies have shapes:

(30%)

2(a) Equation is $EI \dfrac{\partial^4 w}{\partial x^4} = -\rho A \dfrac{\partial^2 w}{\partial t^2}$

Let $w(x,t) = u(x)e^{i\omega t}$ $\rightarrow$ $\dfrac{d^4 u}{dx^4} = \alpha^4 u$, $\alpha^4 = \dfrac{\rho A}{EI}\omega^2$

General solution is $u = A\cos\alpha x + B\sin\alpha x + C\cosh\alpha x + D\sinh\alpha x$

At $x=0$: $\begin{cases} u'' = 0 & \rightarrow \quad -K_1 + K_3 = 0 \\ u''' = 0 & \rightarrow \quad -K_2 + K_4 = 0 \end{cases}$

At $x=L$: $\begin{cases} u'' = 0 \rightarrow & -K_1\cos\alpha L - K_2\sin\alpha L + K_1\cosh\alpha L + K_2\sinh\alpha L = 0 \\ u''' = 0 \rightarrow & K_1\sin\alpha L - K_2\cos\alpha L + K_1\sinh\alpha L + K_2\cosh\alpha L = 0 \end{cases}$

For solution with K_1, K_2 not both zero need 2×2 determinant $= 0$

$\therefore (\cosh\alpha L - \cos\alpha L)^2 - (\sinh\alpha L - \sin\alpha L)(\sinh\alpha L + \sin\alpha L) = 0$

which simplifies to $\cos\alpha L \cosh\alpha L = 1$

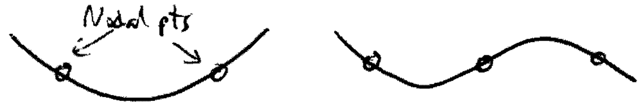
Roots give α and hence ω. (25%)

(b) $f_n = \dfrac{\omega_n}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{EI}{\rho A}} \times \dfrac{z^2}{L^2}$ where z is a root of $\cos z \cosh z = 1$

For a rectangular section, $A = bh$, $I = bh^3/12$ (data book)

So with the given values, the first two frequencies are given by $\dfrac{1}{2\pi}\sqrt{\dfrac{E}{12\rho}}\dfrac{h}{L^2}z^2 = 411, 1132$ Hz

Mode shapes



(25%)

(c) For the weighted beam, $V = \frac{1}{2}EI\int_0^L \phi_n''^2\,dx$

and $\tilde{T} = \frac{1}{2}\rho A\int_0^L \phi_n^2\,dx + \frac{1}{2}M\phi_n^2(a)$

for a mode shape $\phi_n(x)$.

So by Rayleigh, $\Omega_n^2 \approx \dfrac{V}{\tilde{T}}$, while original $\omega_n^2 = \dfrac{V}{\frac{1}{2}\rho A\int\phi_n^2\,dx}$

So $\Omega_n^2 \simeq \omega_n^2\left[1 + \dfrac{M}{\rho A}\dfrac{\phi_n^2(a)}{\int\phi_n^2\,dx}\right]^{-1}$

$\simeq \omega_n^2\left[1 - \dfrac{M}{\rho A}\dfrac{\phi_n^2(a)}{\int\phi_n^2\,dx}\right]$ by binomial

So required $B = -\dfrac{M}{\rho A\int_0^L\phi_n^2\,dx}$ (30%)

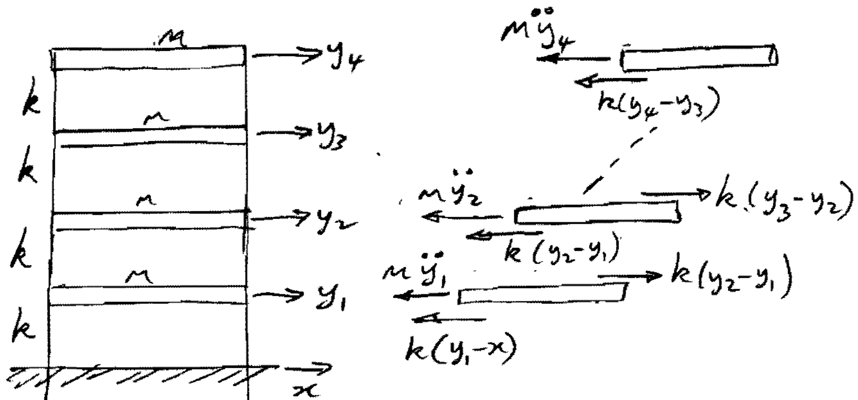
(d) Ratio of frequencies is $\left(\frac{7.85}{4.73}\right)^2 = 2.75$

Nearest whole number is 3. Can only reduce frequencies by added mass, which is why $B<0$.

So need to reduce the frequency of mode 1 while keeping mode 2 fixed. The simplest way to do this is to add mass at the centre of the beam, so that $\phi_n^2(a)$ is large for mode 1 but zero for mode 2 since all antisymmetric modes have a node at the centre.

(20%)

3(a)



d'Alembert: $\sum F = 0$ ←

$$m\ddot{y}_1 + k(y_1 - x) - k(y_2 - y_1) = 0$$

$$m\ddot{y}_2 + k(y_2 - y_1) - k(y_3 - y_2) = 0$$

$$\vdots$$

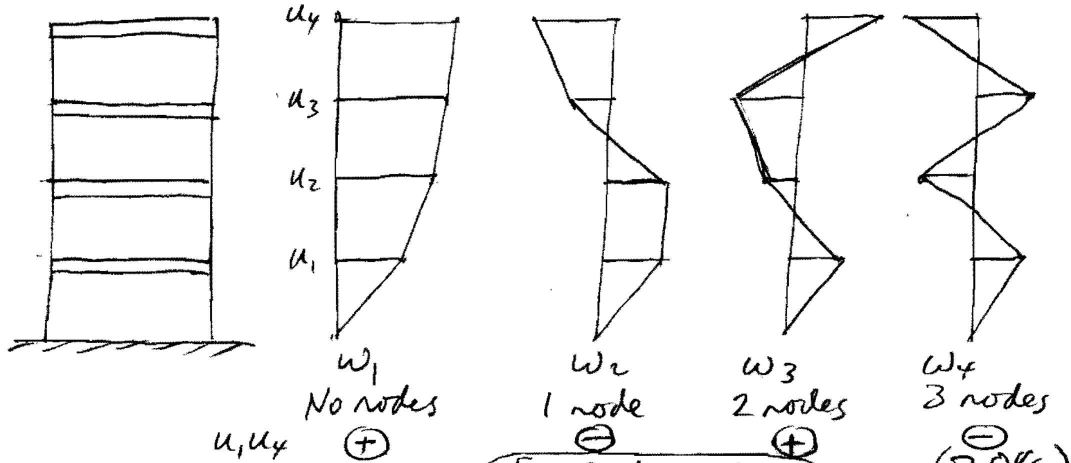
$$m\ddot{y}_4 + k(y_4 - y_3) = 0$$

ie

$$\begin{bmatrix} m & & & \\ & m & & \\ & & m & \\ & & & m \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \\ \ddot{y}_4 \end{Bmatrix} + \begin{bmatrix} 2k & -k & 0 & 0 \\ -k & 2k & -k & 0 \\ 0 & -k & 2k & -k \\ 0 & 0 & -k & k \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \begin{Bmatrix} kx \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

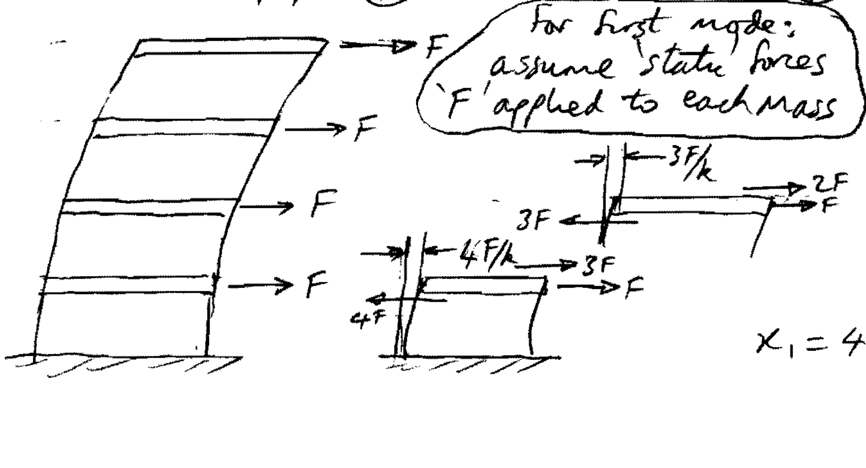
kx term in first row is equivalent to a force applied to mass 1. (20%)

(b)



(20%)

(d)



$x_1 = x_2 + F/k$

$x_3 = x_2 + 2F/k$

$x_2 = x_1 + 3F/k$

$x_1 = 4F/k$

3(cont)

So assumed deflected shape is: $\underline{u} = \frac{F}{k}[4 \quad 7 \quad 9 \quad 10]$

ie assumed mode shape: $\underline{u}^{(1)} = [0.4 \quad 0.7 \quad 0.9 \quad 1.0]$

Rayleigh: $\omega_1^2 = \frac{V_{max}}{T^*} = \frac{\frac{1}{2}k[y_1^2 + (y_2-y_1)^2 + (y_3-y_2)^2 + (y_4-y_3)^2]}{\frac{1}{2}m[y_1^2 + y_2^2 + y_3^2 + y_4^2]}$

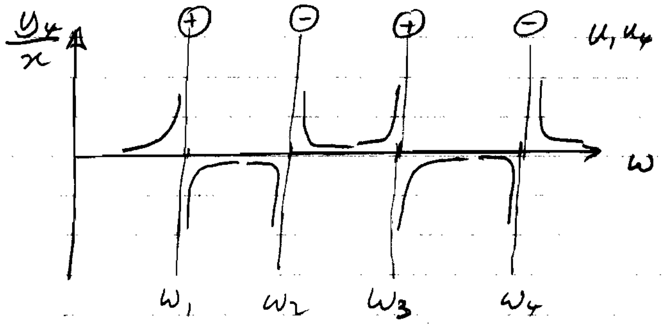
Using $\underline{u}^{(1)}$: $\omega_1^2 = \frac{k}{m}\left[\frac{0.4^2 + 0.3^2 + 0.2^2 + 0.1^2}{0.4^2 + 0.7^2 + 0.9^2 + 1^2}\right] = 0.122\frac{k}{m}$

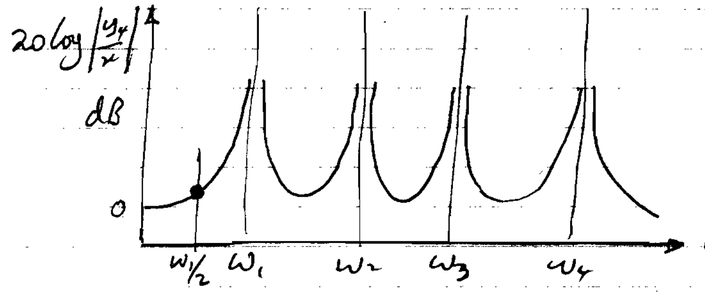
$\therefore \omega_1 \simeq 0.349\sqrt{k/m}$ (c/f $0.1206\,k/m$ exact)

(20%)

(d) Transfer function: $H_{14} = \frac{y_4}{kx} = \sum_n \frac{\hat{u}_1^{(n)}\hat{u}_4^{(n)}}{\omega_n^2 - \omega^2}$

From part (b) $u_1^{(n)}u_4^{(n)}$ alternate sign with n.





← Minima between resonances ie no anti-resonances

(15%)

(e) Estimate $\left|\frac{y_4}{x}\right|$ at $\omega_1/2$ Using $\frac{y_4}{kx} = \frac{\hat{u}_1^{(1)}\hat{u}_4^{(1)}}{\omega_1^2 - (\omega_1/2)^2}$

ie use only first mode (ignore higher modes)

First normalise the assumed mode shape. Assume $\hat{u}_1 = \alpha u_1$

and use $\hat{\underline{u}}_1^T[m]\hat{\underline{u}}_1 = 1$ ie $\alpha^2 \underline{u}_1^T[m]\underline{u}_1 = 1$

So $\alpha^2 [0.4 \quad 0.7 \quad 0.9 \quad 1]\begin{bmatrix} m & & & \\ & m & & \\ & & m & \\ & & & m \end{bmatrix}\begin{Bmatrix} 0.4 \\ 0.7 \\ 0.9 \\ 1 \end{Bmatrix} = 1$

3 (cont) So $m\alpha^2(0.4^2+0.7^2+0.9^2+1^2)=1 \Rightarrow \alpha = \frac{0.638}{\sqrt{m}}$

$\Rightarrow$ Normalized mode shape is: $\hat{\underline{u}}^{(1)} = \frac{1}{\sqrt{m}}[0.255 \quad 0.447 \quad 0.574 \quad 0.638]$

Now $\omega_1^2 = 0.122\, k/m$ (part (c))

So $\omega^2 = \left(\frac{\omega_1}{2}\right)^2 = 0.031\, k/m$

Hence $H_{41} = \frac{y_4}{kx} = \frac{\hat{u}_1^{(1)}\hat{u}_4^{(1)}}{\omega_1^2 - \omega_1^2/4} = \frac{(0.255)(0.638)/m}{0.122\frac{k}{m}(1-\frac{1}{4})}$

$\therefore \left|\frac{y_4}{x}\right| = \underline{\underline{1.79}}$ (ie 5.1 dB)

<u>Assumptions</u>:

(i) Used only first mode: quite accurate at $\omega = \omega_1/2$ ✓

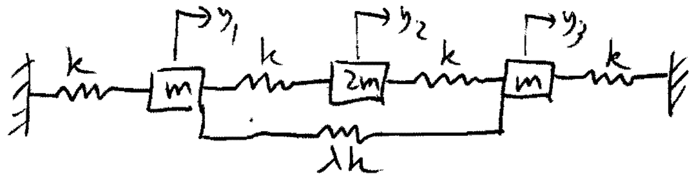
(ii) Ignored damping: OK away from resonance ✓

(iii) Used <u>assumed mode shape</u> instead of <u>eigenvector</u> ?
This is probably the largest source of error

(iv) Usual modelling assumptions: linearity
ignored rotation of masses, bending of floors, etc ✓

(25%)

4 (a)



$$V = \tfrac{1}{2}ky_1^2 + \tfrac{1}{2}k(y_2-y_1)^2 + \tfrac{1}{2}k(y_3-y_2)^2 + \tfrac{1}{2}ky_3^2 + \tfrac{1}{2}\lambda k(y_3-y_1)^2$$

$$\therefore K = k\begin{bmatrix} 2+\lambda & -1 & -\lambda \\ -1 & 2 & -1 \\ -\lambda & -1 & 2+\lambda \end{bmatrix}$$

Mass matrix $M = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

[20%]

(b) All modes must be symmetric or antisymmetric about the central mass. An antisymmetric mode can only take the form $\underline{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, so try this.

Note: "symmetric" and "antisymmetric" are used here to describe the vector: the physical motion is the opposite way round

$$K\underline{u} = k\begin{bmatrix} 2+2\lambda \\ 0 \\ -2-2\lambda \end{bmatrix}, \quad M\underline{u} = m\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

so $K\underline{u} = \omega^2 M\underline{u}$ with $\omega^2 = \dfrac{2k(1+\lambda)}{m}$

A symmetric mode must take the form $\begin{bmatrix} a \\ b \\ a \end{bmatrix}$, so recalculate energy for this: $V = \tfrac{1}{2}ka^2 + \tfrac{1}{2}k(b-a)^2 + \tfrac{1}{2}k(b-a)^2 + \tfrac{1}{2}ka^2$

So 2 DoF stiffness matrix is $K' = k\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}$

Corresponding mass matrix is $M' = m\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

So $|K - \omega^2 M| = 0 \rightarrow (2k - \omega^2 m)(k - \omega^2 m) - k^2 = 0$

so if $\Omega^2 = \omega^2 \dfrac{m}{k}$, $\quad \Omega^4 - 3\Omega^2 + 1 = 0$

$$\therefore \Omega^2 = \tfrac{1}{2}\left(3 \pm \sqrt{9-4}\right) = \tfrac{1}{2}\left(3 \pm \sqrt{5}\right)$$

To get mode shape, note $4ka - 2kb = \omega^2 . 2ma$

$$\therefore 2a - b = \Omega^2 a, \text{ so } b = a(2 - \Omega^2) = \frac{a}{2}(1 \mp \sqrt{5})$$

So in summary: (a) mode $\begin{bmatrix}1\\0\\-1\end{bmatrix}$ with $\omega^2 = 2(1+\lambda)\frac{k}{m}$

(b) mode $\begin{bmatrix}1\\\frac{1}{2}(1+\sqrt{5})\\1\end{bmatrix}$ with $\omega^2 = \frac{1}{2}(3-\sqrt{5})\frac{k}{m}$

(c) mode $\begin{bmatrix}1\\\frac{1}{2}(1-\sqrt{5})\\1\end{bmatrix}$ with $\omega^2 = \frac{1}{2}(3+\sqrt{5})\frac{k}{m}$

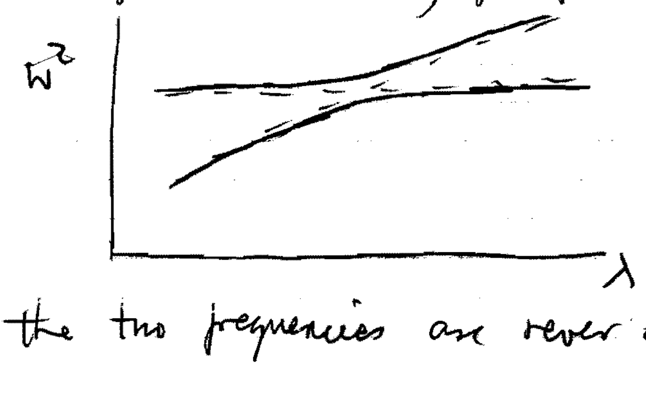
(55%)

(c) Mode (b) is always the lowest frequency.
Modes (a) and (c) can be either way round: increasing λ raises the antisymmetric mode (a) without changing mode (c), while for $\lambda = 0$, mode (c) is at a higher frequency than mode (a).

Modes (a) and (c) have equal frequency when

$$2(1+\lambda) = \frac{1}{2}(3+\sqrt{5}), \text{ ie } \lambda = \frac{3+\sqrt{5}}{4} - 1 = \frac{\sqrt{5}-1}{4}$$

If you tried to build this structure, it is inevitable that perfect symmetry would not be obtained. Once the modes are not exactly symmetric and antisymmetric, a full 3 DoF problem has to be solved. The expected behaviour then will show "curve veering" as in Q1 on Examples Sheet 2. As a function of λ for example, one would expect to see a graph for modes (a) and (c):



so that the two frequencies are never equal.

(25%)