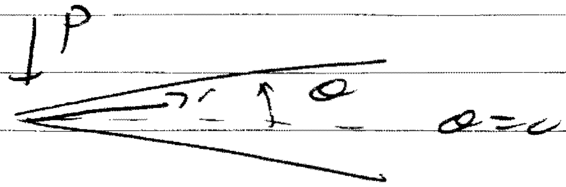
3C7 2013

Q1.

a)



$$\phi = -Cr\theta\cos\theta$$

$$\frac{\partial \phi}{\partial r} = -C\theta\cos\theta$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} = \frac{\partial^2 \phi}{\partial r^2} = 0$$

$$\frac{\partial \phi}{\partial \theta} = -Cr\cos\theta + Cr\theta\sin\theta$$

$$\frac{\partial^2 \phi}{\partial \theta^2} = 2Cr\sin\theta + Cr\theta\cos\theta$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = 0$$

$$\sigma_{rr} = -\frac{C\theta\cos\theta}{r} + \frac{C\theta\cos\theta}{r} + \frac{2C\sin\theta}{r}$$

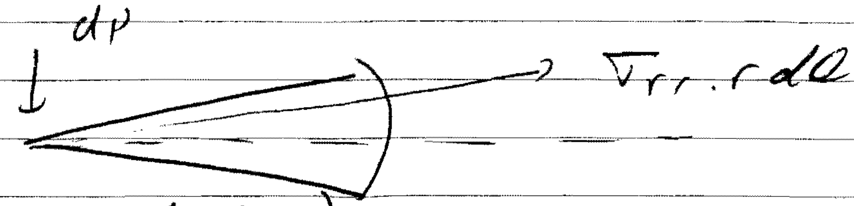
$$= \frac{2C\sin\theta}{r}$$

BC on free edges: $\sigma_{\theta\theta} = 0$ ✓

$\sigma_{r\theta} = 0$ ✓

both satisfied.

Q1 (b) Symmetry : anti-symmetric about $\theta = 0$



Equilibrium : (vertical)

$$dP = \sigma_{rr} \cdot r\,d\theta \sin\theta$$

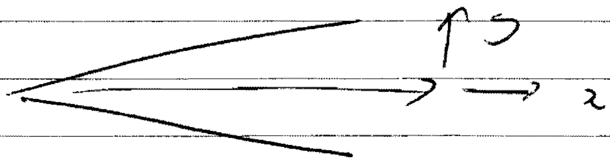
$$= 2C \sin^2\theta\,d\theta$$

Integrate, $\displaystyle P = 2C \int_{-\alpha}^{\alpha} \sin^2\theta\,d\theta$

$$\longrightarrow \quad C = P/(2\alpha - \sin 2\alpha)$$

Horizontal equilibrium satisfied since σ_{rr} anti-symmetric in θ.

(c)

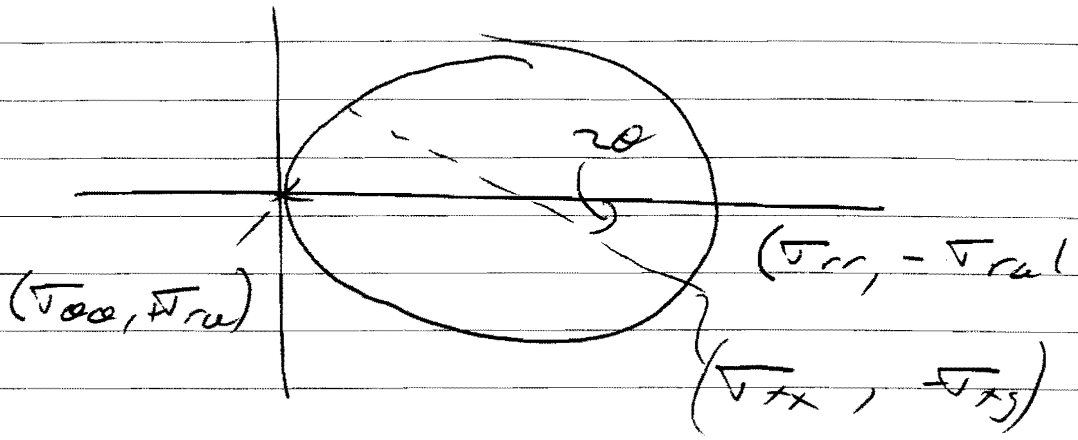


$r = (x^2 + y^2)^{1/2}$, $\sin\theta = y/(x^2+y^2)^{1/2}$

$\theta = \tan^{-1}(y/x)$ | $\cos\theta = x/(x^2+y^2)^{1/2}$.

$$\sigma_{rr} = \frac{2C\,y}{(x^2+y^2)}$$

Mohrs circle



$$\sigma_{xy} = \frac{\sigma_{rr}}{2} \sin 2\theta$$

$$\sin 2\theta = 2 \sin\theta \cos\theta$$

$$\longrightarrow \quad \sigma_{xy} = \frac{2Cxy^2}{(x^2+y^2)^2}$$

- Max sheer stress at top & bottom.

Q2. (a)(i)

Use Lamé

$$\sigma_{rr} = A - \frac{B}{r^2} - \frac{E\alpha}{r^2}\int_0^r rT\,dr$$

$$\sigma_{\theta\theta} = A + \frac{B}{r^2} + \frac{E\alpha}{r^2}\int_0^r rT\,dr - E\alpha T$$

Finite stress @ $r=0 \longrightarrow B=0$

Need $$\frac{E\alpha}{r^2}\int_0^r rT\,dR = \frac{2E\alpha T_0}{r^2}\int_0^r \left(R - \frac{R^3}{D^2}\right)dR$$

$$= E\alpha T_0\left[1 - \frac{1}{2}\left(\frac{r}{D}\right)^2\right].$$

$\sigma_{rr} = 0$ @ $r = D/2$

$$\rightarrow \cancel{\sigma_{rr}} = A = \frac{7}{8}E\alpha T_0$$

$$\sigma_{rr} = -\frac{E\alpha T_0}{8} + \frac{E\alpha T_0}{2}\left(\frac{r}{D}\right)^2$$

$$\sigma_{\theta\theta} = \frac{15}{8}E\alpha T_0 - \frac{E\alpha T_0}{2}\left(\frac{r}{D}\right)^2 - 2E\alpha T_0 + 2E\alpha T_0\left(\frac{r}{D}\right)^2$$

$$= \frac{-E\alpha T_0}{8} + \frac{3E\alpha T_0}{2}\left(\frac{r}{D}\right)^2$$

Q2 (a)(ii)

At $r = 0$ $\quad \sigma_{rr} = \dfrac{-E\alpha T_0}{8}$

$$\sigma_{\theta\theta} = \frac{-E\alpha T_0}{8}$$

At $r = D/2$, $\sigma_{rr} = 0$, $\sigma_{\theta\theta} = \dfrac{2E\alpha T_0}{8}$

Worst case @ $r = D/2$

$$\frac{2E\alpha T_0}{8} = 189\,\text{MPa}$$

$$\text{Safety factor} = \frac{350}{189} = 1.85.$$

B) Find edge expansion in unconstrained case:

$$\varepsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - \nu\sigma_{rr}\right) + \alpha\Delta T$$

At $r = \dfrac{D}{2}$, $\sigma_{rr} = 0$, $\cancel{\Delta} T = \dfrac{3T_0}{2} \Rightarrow \Delta T = \dfrac{T_0}{2}$

$$\varepsilon_{\theta\theta} = \frac{\alpha T_0}{4} + \frac{\cancel{}\alpha T_0}{2} = \frac{\cancel{}\alpha T_0}{\cancel{}}$$

Need to provide change in uniform stress to give equal but opposite strain:

$$\Delta\sigma = \frac{E}{1-\nu^2}\left(-\frac{\cancel{}\alpha T_0}{\cancel{}} - \frac{\cancel{}\alpha T_0 \nu}{\cancel{}}\right)$$

$$= \frac{-E}{1-\nu}\,\frac{\cancel{}\alpha T_0}{\cancel{}}$$

$$\therefore \; \Delta\sigma_{rr} = \cancel{\frac{F}{4(1-\nu)}} - \frac{E\alpha T_0}{(1-\nu)}$$

$$\Delta\sigma_{\theta\theta} = \cancel{\frac{F}{4(1-\nu)}} - \frac{E\alpha T_0}{(1-\nu)}$$

Does not change $\sigma_{\theta\theta} - \sigma_{rr}$

Consider $r = 0$ (worst case compressive stress)

$$\sigma_{rr} - \sigma_{\theta\theta} = -\frac{E\alpha T_0}{8(1-\nu)}\left[1-\nu+8\right]$$

$$= \frac{-E\alpha T_0\,(9-\nu)}{8\,(1-\nu)} = \sigma_{rr} - \underset{\uparrow\,0}{\sigma_{33}}$$

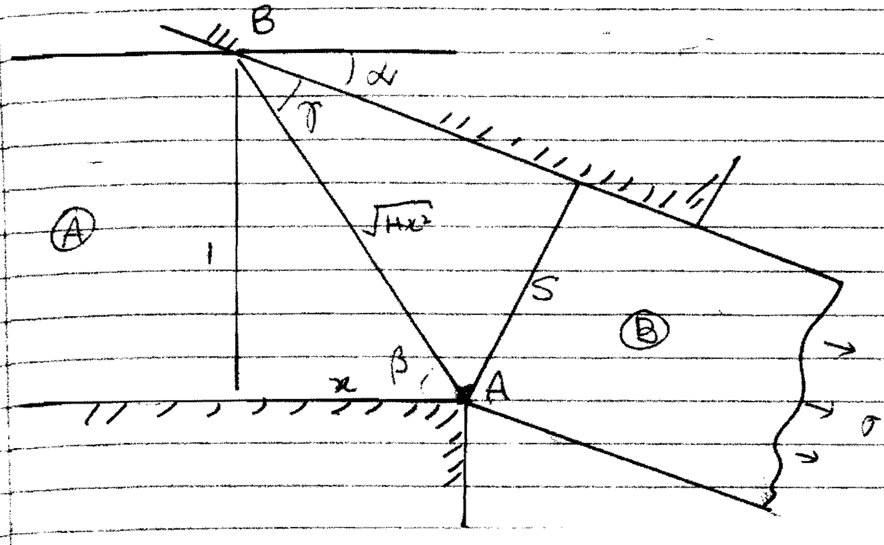
$\longrightarrow$ ~~1984.8 MPa~~ 1310 MPa

Safety factor $\cancel{\frac{350}{1984.5} = 0.17}$

$$\frac{350}{1310} = 0.26$$

NOT SAFE!

Q3



(a) Consider angles at B $\left(\frac{\pi}{2}-\beta\right)+\gamma+\alpha=\frac{\pi}{2}$

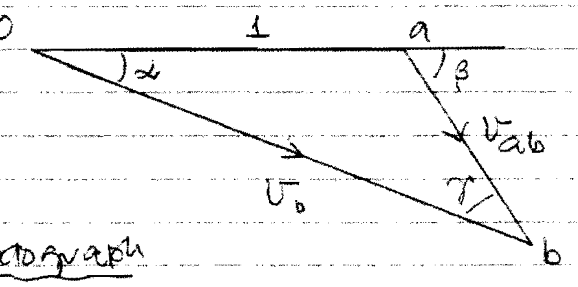
$$\alpha=\beta-\gamma \qquad \gamma=\beta-\alpha$$

$$\sin\alpha=\sin(\beta-\gamma)=\sin\beta\cos\gamma-\cos\beta\sin\gamma$$

But $\sin\beta=\frac{1}{\sqrt{1+x^2}}$; $\cos\beta=\frac{x}{\sqrt{1+x^2}}$ and $\sin\gamma=\frac{s}{\sqrt{1+x^2}}$; $\cos\gamma=\frac{\sqrt{1+x^2-s^2}}{\sqrt{1+x^2}}$

$$\sin\alpha=\frac{\sqrt{1+x^2-s^2}-xs}{1+x^2}$$

(b)



hodograph

Upper Bound given by

$$\sigma.s.v_b=k.l_{ab}.v_{ab}$$

from hodograph

$$\frac{v_b}{\sin(\pi-\beta)}=\frac{v_{ab}}{\sin\alpha}$$

i.e. $v_b=\frac{\sin\beta}{\sin\alpha}v_{ab}$

$\beta=\alpha+\gamma$
$\gamma=\beta-\alpha$

$$\sigma.s\frac{\sin\beta}{\sin\alpha}.\cancel{v_{ab}}=k.\frac{1}{\sin\beta}\cancel{v_{ab}}$$

$$\therefore\ \frac{\sigma s}{k}=\frac{\sin\alpha}{\sin^2\beta}=\frac{\sqrt{1+x^2-s^2}-xs}{1+x^2}\cdot\frac{1+x^2}{1}$$

i.e. $(\sigma' s / k) = \sqrt{1 + x^2 - s^2} - xs$

~~Stroke~~

(c) Now minimise σ' for given s

$$\frac{d(\sigma s/k)}{dx} = \frac{1}{2}(1 + x^2 - s^2)^{-1/2} \cdot 2x - s = 0$$

when $x(1 + x^2 - s^2)^{-1/2} = s$

i.e. $x = s(1 + x^2 - s^2)^{1/2}$

or $x^2 = s^2(1 + x^2 - s^2)$

$$x^2(1 - \cancel{s^2}) = s^2(1 - \cancel{s^2})$$

i.e. $x = s$

then $\sin\alpha = \dfrac{\sqrt{1 + s^2 - s^2} - s^2}{1 + s^2} = \dfrac{1 - s^2}{1 + s^2}$

(d) we have $\dfrac{\sigma}{k} = \dfrac{\sqrt{1 + x^2 - s^2} - xs}{s}$

So if $x = s$ $\dfrac{\sigma}{k} = \dfrac{1 - s^2}{s}$

By $\sigma = 2k$ then $\dfrac{1 - s^2}{s} = 2$

in limit i.e. $1 - s^2 = 2s$

or $s^2 + 2s - 1 = 0$

$\therefore$ $s = \dfrac{-2 \pm \sqrt{8}}{2}$ i.e. $s = \sqrt{2} - 1 = 0.414$

i.e. reduction of area $\Rightarrow 1 - s = 58.6\%$

When $\alpha = \dfrac{1 - (\sqrt{2} - 1)^2}{1 + (\sqrt{2} - 1)^2} = \dfrac{1 - 2 + 2\sqrt{2} - 1}{1 + 2 - 2\sqrt{2} + 1} = \dfrac{-2 + 2\sqrt{2}}{4 - 2\sqrt{2}}$ $\therefore$ $\alpha = 45°$

$\sin\beta = \dfrac{1}{(4 - 2\sqrt{2})^{1/2}} = 67.5°$

Q4

(a) $(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2+2\sigma_t(\rho-1)(\sigma_1+\sigma_2+\sigma_3)=2\rho\sigma_t^2$

If $\rho=3$ and conditions are plane stress i.e. $\sigma_3=0$

$$(\sigma_1-\sigma_2)^2+\sigma_2^2+\sigma_1^2+4(\sigma_1+\sigma_2)\sigma_t=6\sigma_t^2$$

If $\sigma_1=Y$, $\sigma_2=0$ then

$$Y^2+Y^2+4Y\sigma_t=6\sigma_t^2$$

ie $Y^2+2Y\sigma_t=3\sigma_t^2$

So satisfied by $Y=\sigma_t$

hence Stassi condition becomes

$$(\sigma_1-\sigma_2)^2+\sigma_2^2+\sigma_1^2+4Y(\sigma_1+\sigma_2)=6Y^2$$

Von Mises for plane stress would be

$$(\sigma_1-\sigma_2)^2+\sigma_2^2+\sigma_1^2=2Y^2$$

(b) In simple compression (say) $\sigma_2=-\sigma$, $\sigma_1=0$

∴ Stassi $2\sigma^2-4Y\sigma=6Y^2$

letting $\sigma=\sigma/Y$ $\quad \sigma^2-2\sigma-3=0$

So that $|\sigma|=\dfrac{2\pm\sqrt{4+12}}{2}=1\pm2$ ie $\dfrac{\sigma}{Y}=-3$

Von Mises $\quad 2\sigma^2=2Y^2 \quad \dfrac{\sigma}{Y}=-1$

Equal biaxial tension $\sigma_1=\sigma_2=\sigma$ say

Von Mises $\quad 2\sigma^2=2Y^2 \quad \sigma/Y=1$

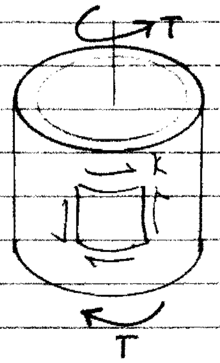
$\dfrac{\sqrt{28}-4}{2}$

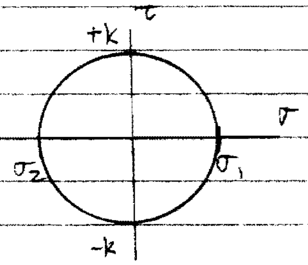
Stassi $\quad \sigma^2+\sigma^2+8\sigma=6$

ie $\quad \sigma^2+4\sigma-3=0 \quad |\sigma|=\dfrac{-4\pm\sqrt{16+12}}{2}$

$\dfrac{|\sigma|}{Y}=0.645$

(c) thin walled tube in torsion





Mohr's circle

$\sigma_1 = k, \ \sigma_2 = -k$

Von Mises

$$4k^2 + k^2 + k^2 = 2Y^2$$

$$\therefore \ k = \frac{Y}{\sqrt{3}}$$

i.e. $\frac{\sigma_1}{Y} = \frac{1}{\sqrt{3}} \ ; \ \frac{\sigma_2}{Y} = -\frac{1}{\sqrt{3}}$

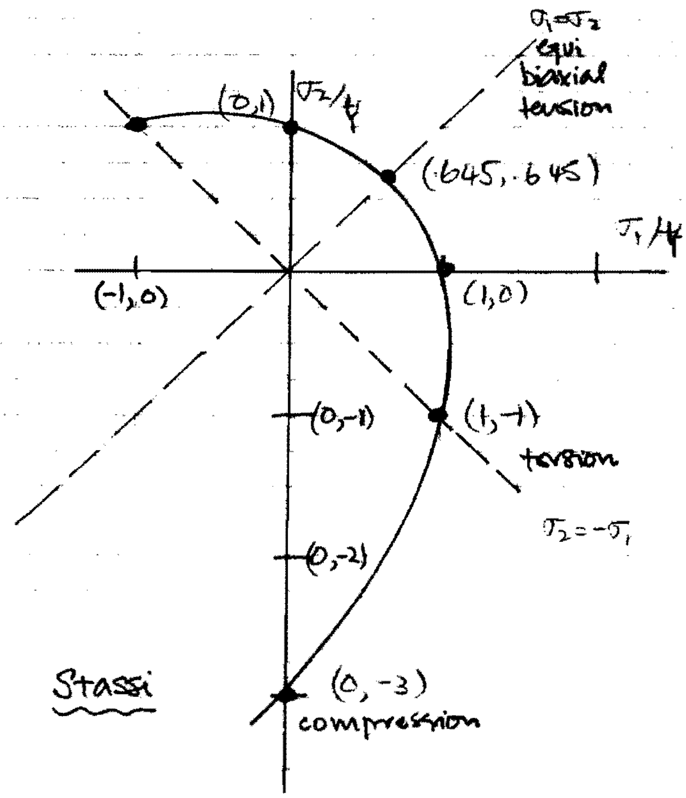
Stassi

$$4k^2 + k^2 + k^2 = 6Y^2$$

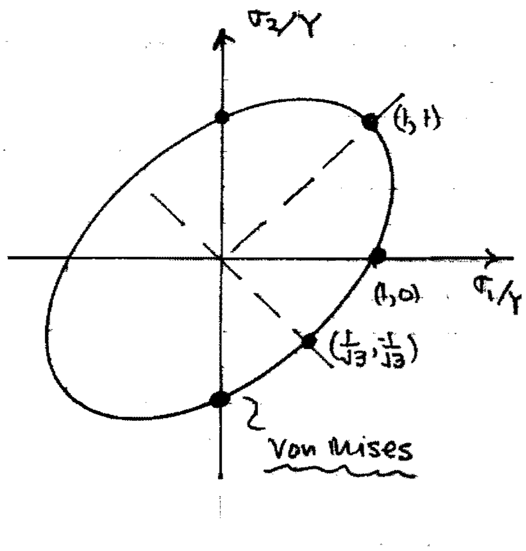
$$\therefore \ k = Y$$

$\therefore \ \frac{\sigma_1}{Y} = 1 \ ; \ \frac{\sigma_2}{Y} = -1$

(d) Comparison of Stassi & von Mises yield loci in principal stress axes shown below



Stassi



Von Mises