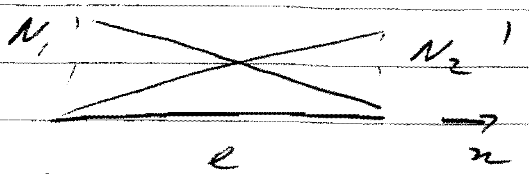
3D7 2013

Q.1

a) $$\underline{\underline{m}}_e = \int_0^l \rho \, \underline{N}^T \underline{N} \, dx$$



$$\underline{N} = \left[-x/l + 1 \quad x/l \right]$$

$$= \rho l \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}$$

$$\underline{\underline{K}}_e = \int_0^l E \underline{B}^T \underline{B} \, dx , \qquad \underline{B} = \left[-1/l \quad 1/l \right]$$

$$E = E_0 + \beta x$$

$$= \int_0^l (E_0 + \beta x) \begin{bmatrix} 1/l^2 & -1/l^2 \\ -1/l^2 & 1/l^2 \end{bmatrix} dx$$

$$= \begin{bmatrix} \frac{E_0}{l} + \beta/2 & -(E_0/l + \beta/2) \\ -(E_0/l + \beta/2) & E_0/l + \beta/2 \end{bmatrix}$$

b)
$$a_{n+1} - a_n = \Delta t \, \dot{a}_{n+1}$$
$$\dot{a}_{n+1} - \dot{a}_n = \Delta t \, \ddot{a}_{n+1}$$

$$\rightarrow \ddot{a}_{n+1} = 1/\Delta t \left(\dot{a}_{n+1} - \dot{a}_n \right)$$

$$= \frac{1}{\Delta t}\left(\frac{1}{\Delta t}\left(a_{n+1} - a_n \right) - \dot{a}_n \right)$$

Insert into $\underline{\underline{M}} \, \ddot{\underline{a}}_{n+1} + \underline{\underline{K}} \, \underline{a}_{n+1} = \underline{f}_{n+1}$

$$\rightarrow \left(\underline{\underline{M}} + \Delta t^2 \underline{\underline{K}} \right) \underline{a}_{n+1} = \Delta t^2 \underline{f}_{n+1} + \underline{\underline{M}} \, \underline{a}_n + \Delta t \, \underline{\underline{M}} \, \dot{\underline{a}}_n$$

Q1 c) Setting $v = \frac{\partial u}{\partial t}$:

$$\rightarrow \quad \left.\begin{aligned} \rho \frac{\partial v}{\partial t} - \frac{\partial}{\partial x}\left(E \frac{\partial u}{\partial x}\right) &= f \quad (1) \\ v - \frac{\partial u}{\partial t} &= 0 \quad (2) \end{aligned}\right\}$$

'nodal' velocity

'nodal' displacement

$$(1) \quad \underline{\underline{M}}\,\underline{\dot{v}} + \underline{\underline{K}}\,\underline{a} = \underline{f}$$

$$(2) \quad \underline{v} - \underline{\dot{a}} = \underline{0}$$

Apply time stepping scheme,

$$\underline{\underline{M}}\,\underline{v}_{n+1} + \Delta t\,\underline{\underline{K}}\,\underline{a}_{n+1} = \Delta t\,\underline{f}_{n+1} + \underline{\underline{M}}\,\underline{v}_n$$

$$\Delta t\,\underline{v}_{n+1} - \underline{a}_{n+1} = -\underline{a}_n$$

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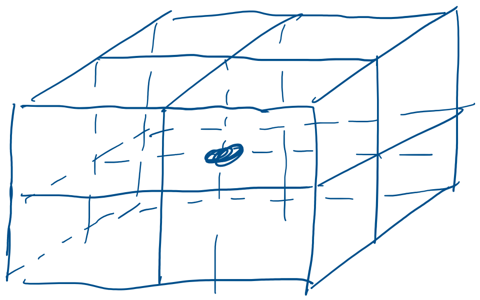
Q2 a) Count number of 'connections' for a node surrounded by cells:



$\Rightarrow$ 27 connections per row (same for both meshes)

b) Double number of cells $\Rightarrow$ doubles the assembly time

Double number of cells $\Rightarrow$ double the number of degrees of freedom

Storage (memory) requirements are $27n$, when n is the number of degrees of freedom

$\Rightarrow$ storage cost doubles

c) LU cost is $O(n^2) \to O(n^3)$, when n is the number of degrees of freedom.

MG cost is $O(n)$

$\Rightarrow$ LU cost increase is betw. $\underset{(2^2)}{4}$ and $\underset{(2^3)}{8}$

$\Rightarrow$ MG cost increase is order 2

d) Consider change in cell edge size:

$h \sim 1/n_V$, where n_V is the number of nodes in each direction

$$n_{cells} = n_V^3 \implies n_V = \sqrt[3]{n_{cells}}$$

Case A: $n_V^{(1)} = \sqrt[3]{n_{cells}}$

Case B: $n_V^{(2)} = \sqrt[3]{2 n_{cells}} = \sqrt[3]{2}\,\sqrt[3]{n_{cells}}$

$\therefore$ doubling number of cells $\implies \sqrt[3]{2} \approx \underline{1.26 \text{ increase}}$ in number of nodes in each direction

$\therefore$ doubling number of cells $\implies 1/\sqrt[3]{2}$ reduction in cell edge length

Order of accuracy is $O(h^2)$

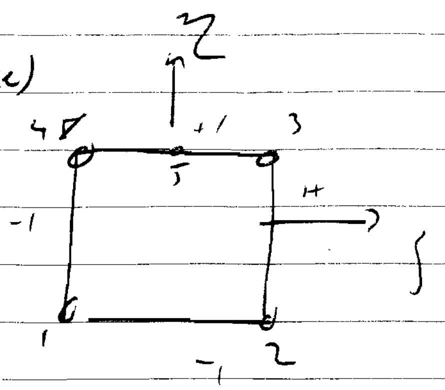
$\implies$ reduction is $\left(\sqrt[3]{2}\right)^2 = 2^{2/3} \approx \underline{1.59}$ factor reduction

e) Critical time step for heat equation scales with h (cell edge length)

$\implies$ time step must be $\underline{\underline{2^{2/3}}}$ times smaller

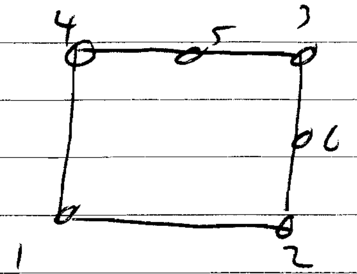
No. Assembly cost per step is double, and time step reduction factor is greater than 1.

Q3 a) (i)



$$N_3 = \tfrac{1}{2}(\xi^2+\xi)\,\tfrac{1}{2}(1+\eta)$$

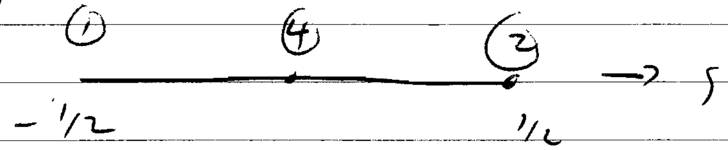
(ii)



$$N_4 = \tfrac{1}{4}(1+\eta)(\xi^2-\xi)$$

$$N_3 = \tfrac{1}{4}(\xi^2+\xi)(\eta^2+\eta)$$

b) (i)



$$N_1 = 2\xi^2-\xi$$

$$N_4 = -4\xi^2+1$$

$$\longrightarrow \bar{q}\int_{-1/2}^{1/2}(2\xi^2-\xi)\,d\xi = \bar{q}\left[\tfrac{2}{3}\xi^3 - \tfrac{\xi^2}{2}\right]_{-1/2}^{1/2}$$

$$= \bar{q}\left(\tfrac{2}{3}\cdot\tfrac{1}{8}\cdot 2\right) = \frac{\bar{q}}{6}$$

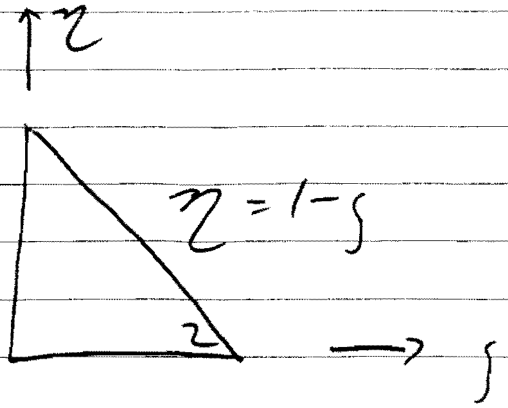
$$\bar{q}\int_{-1/2}^{-1/2}(-4\xi^2+1)\,d\xi = \bar{q}\left[-\tfrac{4}{3}\xi^3+\xi\right]_{-1/2}^{1/2}$$

$$= \bar{q}\,\frac{2}{3}$$

∴ Element flux vector

$$= \left(\bar{q}/6,\ \bar{q}/6;\ 0,\ \frac{2\bar{q}}{3},\ 0,\ 0\right).$$

Q3 (bii)



$$N_2 = 2\xi^2 - \xi$$

$$S\int_0^1\int_0^{1-\xi}(2\xi^2-\xi)\,d\eta\,d\xi = S\int_0^1(2\xi^2-\xi)(1-\xi)\,d\xi$$

$$= S\int(2\xi^2 - 2\xi^3 - \xi + \xi^2)\,d\xi$$

$$= S\left[\xi^3 - \frac{\xi^4}{2} - \frac{\xi^2}{2}\right]_0^1 = 0$$

Q4 (a)(i) $x = 5N_2 + 6.5N_3$

$y = 5N_4 + 7N_3$

$$N_2 = (1+\xi)(1-\eta)/4$$

$$\frac{\partial N_2}{\partial \xi} = (1-\eta)/4 \qquad \frac{\partial N_2}{\partial \eta} = -(1+\xi)/4$$

$$N_3 = (1+\xi)(1+\eta)/4$$

$$\frac{\partial N_3}{\partial \xi} = (1+\eta)/4 \qquad \frac{\partial N_3}{\partial \eta} = (1+\xi)/4$$

$$N_4 = (1-\xi)(1+\eta)/4$$

$$\frac{\partial N_4}{\partial \xi} = -(1+\eta)/4 \qquad \frac{\partial N_4}{\partial \eta} \; (1-\xi)/4$$

$$\frac{\partial x}{\partial \xi} = (5)(0.1057) + (6.5)(0.394) = 3.09$$

$$\frac{\partial x}{\partial \eta} = -(5.0)(0.394) + (6.5)(0.394) = 0.591$$

$$\frac{\partial y}{\partial \xi} = -(5)(0.394) + (7)(0.394) = 0.788$$

$$\frac{\partial y}{\partial \eta} = (5)(0.1057) + (7)(0.394) = 3.287$$

$$\rightarrow \quad \underline{\underline{J}} = \begin{bmatrix} 3.09 & 0.788 \\ 0.591 & 3.287 \end{bmatrix}.$$

Q4 (a)(iv) $\underline{\underline{J}}^{-1} = \begin{bmatrix} 0.3392 & -0.081 \\ -0.061 & 0.31885 \end{bmatrix}$.

$$\frac{\partial N_3}{\partial x} = \frac{\partial N_3}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial N_3}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}$$

$$= (0.3943)(0.3392) + (0.3943)(-0.081)$$
$$= 0.102$$

$$\frac{\partial N_3}{\partial y} = \frac{\partial N_3}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial N_3}{\partial \eta} \frac{\partial \eta}{\partial y}$$

$$= (0.3943)(-0.061) + (0.3943)(0.31885)$$
$$= 0.102$$

(b) $$k \int_\Omega \left(\frac{\partial^2 T}{\partial x} + \frac{\partial^2 T}{\partial y^2} + s \right) w \, d\Omega = 0$$

Integrate by parts,

$$-k \int \left(\frac{\partial T}{\partial x} \frac{\partial w}{\partial x} + \frac{\partial T}{\partial y} \frac{\partial w}{\partial y} \right) d\Omega$$

$$+ \cancel{k} \underbrace{\int_{\partial\Omega} k \left(\frac{\partial T}{\partial x} n_x + \frac{\partial T}{\partial y} n_y \right) w \, d\Gamma} + \int_\Omega s w \, d\Omega = 0$$

replace this with boundary condition, eg. $\bar{q} - \beta(T - T_\infty)$ $\longrightarrow$ weak form.