Engineering Tripos Part IIA, Module 3F1, SIGNALS AND SYSTEMS SAMPLE SOLUTIONS TO EXAM MAY 2013

1. **Solution:** *A discrete-time system with input sequence $\{u_k\}$ and output sequence $\{y_k\}$, has transfer function* $G(z) = \dfrac{(z+1)}{z(z-1)}$.

(a) *Calculate the step response of this system, and check your answer is consistent with the initial value theorem and/or the final value theorem if either one applies.*

$U(z) = z/(z-1)$ and

$$Y(z) = G(z)U(z) = \frac{(z+1)}{(z-1)^2} = \frac{(z^{-1}+z^{-2})}{(1-z^{-1})^2} \quad \checkmark$$

and from the data book z-transform table we have $\mathcal{Z}\{k\}_{k\geq 0} = \frac{z^{-1}}{(1-z^{-1})^2}$ and $\mathcal{Z}\{k-1\}_{k\geq 1} = \frac{z^{-2}}{(1-z^{-1})^2}$ and hence $y_0 = 0$ and $y_k = 2k-1$ for $k \geq 1$. The final value theorem does not apply since the poles of $(z-1)Y(z)$ include one at $z = 1$, but the initial value theorem does apply and $y_0 = \lim_{z\to\infty} Y(z) = 0$.

(b) *Write down a difference equation with this transfer function and check the first three values of the step response calculated above agree with the corresponding solution of the difference equation.*

Difference equation,

$$y_{k+2} - y_{k+1} = u_{k+1} + u_k$$

has transfer function $G(z)$. With $y_k = 0$ for $k < 0$ and $u_k = 1$ for $k \geq 0$ successively solving this difference equation gives $y_0 = 0$, $y_1 = 1$ and $y_2 = y_1 + u_1 + u_0 = 3$, agreeing with the above.

(c) *Show that*

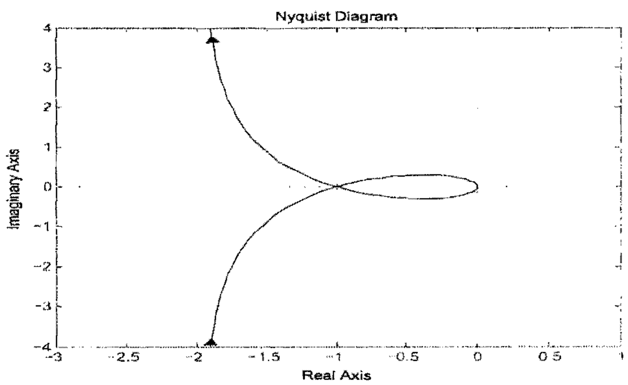
$$G(e^{j\theta}) = \frac{-j}{\tan(\theta/2)}\, e^{-j\theta}.$$

Note that $\frac{e^{j\theta}+1}{e^{j\theta}-1} = \frac{e^{j\theta/2}+e^{-j\theta/2}}{e^{j\theta}-e^{-j\theta/2}} = \frac{2\cos(\theta/2)}{2j\sin(\theta/2)}$ which immediately gives the result.

(d) *Sketch the Nyquist diagram for $G(z)$. This system is now connected in a standard unity gain negative feedback arrangement with a precompensator with a constant gain, K. Using the Nyquist stability criterion determine what values of K give closed-loop stability.*

$$
\begin{aligned}
G(e^{j\theta}) &= \frac{-j}{\tan(\theta/2)}\, e^{-j\theta} \\
&= -\frac{\sin(\theta)}{\tan(\theta/2)} - \frac{j\cos(\theta)}{\tan(\theta/2)} \\
&= -2\cos^2(\theta/2) - \frac{j\cos(\theta)}{\tan(\theta/2)} \\
&\rightarrow -2 - j\infty \quad \text{as } \theta \rightarrow 0
\end{aligned}
$$

Note that at $G(e^{j\pi/2}) = -1$. This gives the following Nyquist plot, where the semicircle goes round to the right. There will hence be no encirclements of the point $-1/K$ if $0 < K < 1$ when the closed-loop system will be stable. There will be two clockwise encirclements if $K > 0$ and one clockwise encirclement if $K < 0$.



(e) *For $K = 1/2$ and the external reference signal, $r_k = \cos(\omega kT)$, determine the behaviour of the error, $e_k = r_k - y_k$ as k becomes large for the three cases, $\omega T = 0,\ \pi/4,\ \pi$.*

$E\cancel{Y}(z) = \frac{1}{1+KG(z)}R(z)$ and since $K = 1/2$ is stabilising we have that $\overset{e}{\cancel{y_k}} \to |H(e^{j\theta})| \cos(\omega kT + \angle G(e^{j\theta}))$, where $H(z) = \frac{1}{1+KG(z)}$, $\theta = \omega T = 0,\ \pi/4,\ \pi$. Now

(i) since $G(e^{j0}) = \infty$, $H(e^{j0}) = 0$, so $e_k \to 0$;

(ii) $G(e^{j\pi}) = 0$, $H(e^{j\pi}) = 1$, so $e_k \to (-1)^k$;

(iii) $G(e^{j\pi/4}) = \frac{-j\cos(\pi/4)}{\tan(\pi/8)} - \frac{\sin(\pi/4)}{\tan(\pi/8)} = -1.71 - 1.71j$, $H(e^{j\pi/4}) = \frac{1}{0.146-0.854j} = 1.15e^{1.40j}$, so $e_k \to 1.15\cos(\omega kT + 1.40)$

(f) *If the input to $G(z)$ is $u_k = (-1)^k$ for $k \geq 0$, calculate the output and comment on its relation to the frequency response.*

In this case $U(z) = \frac{z}{z+1}$ so $Y(z) = \frac{1}{z-1}$ and hence $y_0 = 0$ and $y_k = 1$ for $k \geq 1$. Hence the output due to the sinusoidal input is zero because $G(e^{j\pi}) = 0$ but the output does not tend to zero because the system is not stable due to the pole at $z = 1$.

2. **Solution:**

(a) *A linear discrete-time system with input sequence $\{u_k\}$ and output sequence $\{y_k\}$, has pulse response sequence $\{g_k\}$ and transfer function $G(z)$.*

i. *Show that if*

$$\sum_{k=0}^{\infty} |g_k| = M < \infty \tag{1}$$

then bounded inputs will produce bounded outputs.

Standard bookwork.

ii. *In the case $G(z) = \dfrac{1}{(z^2+1)}$ show that (1) does not hold and that there exists a bounded input that gives an unbounded output sequence.*

For $G(z) = \frac{1}{z^2+1}$, the pulse response will be $g_0 = 0, g_1 = 0, g_2 = 1, g_3 = 0, g_4 = -1, g_5 = 0, g_6 = 1, \cdots$. Or from the databook with $\omega_0 T = \pi/2$, $r = 1$, $a = 0$, giving $g_k = \sin(\pi(k-1)/2)$, for $k \geq 2$. Clearly, although the g_k remain bounded the $\sum |g_k| \to \infty$. If the input sequence, $u_k = g_k$ then $Y(z) = \frac{1}{(z^2+1)^2} = z^{-2}(1+z^{-2})^{-2} = z^{-2}(1 - 2z^{-2} + 3z^{-4} - 4z^{-6} + \cdots (-1)^k(k+1)z^{-2k} + \cdots$. This clearly increases without bound as $k \to \infty$, giving an unbounded output for a bounded input.

Engineering Tripos Part IIA, Module 3F1,
SIGNALS AND SYSTEMS
SAMPLE SOLUTIONS TO EXAM MAY 2013

1. **Solution:** *A discrete-time system with input sequence $\{u_k\}$ and output sequence $\{y_k\}$, has transfer function $G(z) = \dfrac{(z+1)}{z(z-1)}$.*

(a) *Calculate the step response of this system, and check your answer is consistent with the initial value theorem and/or the final value theorem if either one applies.*

$U(z) = z/(z-1)$ and

$$Y(z) = G(z)U(z) = \frac{(z+1)}{(z-1)^2} = \frac{(z^{-1} + z^{-2})}{(1-z^{-1})^2}$$

and from the data book z-transform table we have $\mathcal{Z}\{k\}_{k\geq 0} = \frac{z^{-1}}{(1-z^{-1})^2}$ and $\mathcal{Z}\{k-1\}_{k\geq 1} = \frac{z^{-2}}{(1-z^{-1})^2}$ and hence $y_0 = 0$ and $y_k = 2k-1$ for $k \geq 1$. The final value theorem does not apply since the poles of $(z-1)Y(z)$ include one at $z = 1$, but the initial value theorem does apply and $y_0 = \lim_{z\to\infty} Y(z) = 0$.

(b) *Write down a difference equation with this transfer function and check the first three values of the step response calculated above agree with the corresponding solution of the difference equation.*

Difference equation,

$$y_{k+2} - y_{k+1} = u_{k+1} + u_k$$

has transfer function $G(z)$. With $y_k = 0$ for $k < 0$ and $u_k = 1$ for $k \geq 0$ successively solving this difference equation gives $y_0 = 0$, $y_1 = 1$ and $y_2 = y_1 + u_1 + u_0 = 3$, agreeing with the above.

(c) *Show that*

$$G(e^{j\theta}) = \frac{-j}{\tan(\theta/2)}\, e^{-j\theta}.$$

Note that $\frac{e^{j\theta}+1}{e^{j\theta}-1} = \frac{e^{j\theta/2}+e^{-j\theta/2}}{e^{j\theta}-e^{-j\theta/2}} = \frac{2\cos(\theta/2)}{2j\sin(\theta/2)}$ which immediately gives the result.

(d) *Sketch the Nyquist diagram for $G(z)$. This system is now connected in a standard unity gain negative feedback arrangement with a precompensator with a constant gain, K. Using the Nyquist stability criterion determine what values of K give closed-loop stability.*

2.(b)

$$X(t) \longrightarrow \boxed{\begin{array}{c} h(t) \\ H(\omega) \end{array}} \longrightarrow Y(t)$$

(i) $r_{XX}(\tau) = E\left[X(t)\,X(t+\tau)\right]$

$Y(t) = X(t) * h(t)$ (convolution)

$$= \int_{-\infty}^{\infty} h(\beta)\, X(t-\beta)\, d\beta$$

$$r_{XY}(\tau) = E\left[X(t)\,Y(t+\tau)\right]$$

$$= E\left[X(t)\int_{-\infty}^{\infty} h(\beta)\, X(t+\tau-\beta)\, d\beta\right]$$

$$= \int_{-\infty}^{\infty} h(\beta)\, E\left[X(t)\,X(t+\tau-\beta)\right] d\beta$$

$$= \int_{-\infty}^{\infty} h(\beta)\, r_{XX}(\tau-\beta)\, d\beta$$

$= \underline{\underline{h(\tau) * r_{XX}(\tau)}}$ (convolution)

2 b. (ii) $S_x(\omega)$ is the Fourier transform of $r_{xx}(\tau)$.

$S_{xy}(\omega)$ is the Fourier transform of $r_{xy}(\tau)$

(iii) Taking the Fourier transform of the convolution result in b(i):

$$S_{xy}(\omega) = H(\omega) \cdot S_{xx}(\omega)$$

$$\therefore H(\omega) = \frac{S_{xy}(\omega)}{S_{xx}(\omega)} = \frac{F.T.\{r_{xy}(\tau)\}}{F.T.\{r_{xx}(\tau)\}}$$

In this way $H(\omega)$ may be calculated when the input to the system comprises random perturbations which typically occur in normal operation (eg an airliner cruising at altitude, & subject to atmospheric turbulence, or a power station subject to random load fluctuations)

3. (a) $\Phi_X(u) = E[e^{juX}]$

$$= \int_{-\infty}^{\infty} e^{jux} f_X(x)\, dx$$

whereas the Fourier transform of $f_X(t)$ is

$$F_X(\omega) = \int_{-\infty}^{\infty} f_X(t)\, e^{-j\omega t}\, dt$$

Hence exchanging x for t, & u for $(-\omega)$, the two formulae are equivalent.

So the characteristic function $\Phi_X(u) = F_X(-u)$ where $F_X(\cdot)$ is the Fourier transform of the pdf, $f_X(x)$.

(b) $Y = X_1 + X_2$

Now joint pdf $f(y, x_1) = f(y|x_1) \cdot f_1(x_1)$

Now $f(y|x_1) = f_2(y - x_1)$

& marginalising out x_1 gives

$$f(y) = \int_{-\infty}^{\infty} f(y|x_1)\, f_1(x_1)\, dx_1$$

$$= \int_{-\infty}^{\infty} f_2(y - x_1)\, f_1(x_1)\, dx_1$$

$$= f_2 * f_1 \quad \text{(convolution)}$$

3 (b - cont.)

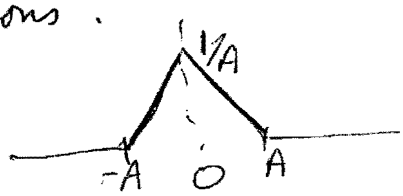
Taking Fourier transforms

$$F_y(\omega) = F_{X_2}(\omega) \cdot F_{X_1}(\omega)$$

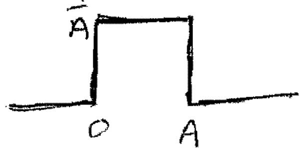
& replacing ω with $-u$.

$$\Phi_y(u) = \Phi_{X_2}(u) \cdot \Phi_{X_1}(u)$$

Hence adding X_1 & X_2 results in multiplication of their characteristic functions.

(c) If $f_1(x)$ is 

then its F.T. is $\text{sinc}^2\left(\frac{\omega A}{2}\right)$ (from Data Book)

and if $f_2(x)$ is 

then its F.T is $\text{sinc}\left(\frac{\omega A}{2}\right) \cdot e^{-j\omega A/2}$

Letting $u = -\omega$:

$$\therefore \Phi_{X_1}(u) = \text{sinc}^2\left(\frac{-uA}{2}\right) = \text{sinc}^2\left(\frac{uA}{2}\right)$$

$$\& \quad \Phi_{X_2}(u) = \text{sinc}\left(\frac{-uA}{2}\right) \cdot e^{juA/2} = \text{sinc}\left(\frac{uA}{2}\right) \cdot e^{juA/2}$$

$$\therefore \Phi_y(u) = \underline{\underline{\text{sinc}^3\left(\frac{uA}{2}\right) \cdot e^{juA/2}}}$$

3. (d) Since $\Phi_Y(u) = \int_{-\infty}^{\infty} f_Y(y)\, e^{juy}\, dy$

$$\frac{d}{du}\Phi_Y(u) = \int_{-\infty}^{\infty} jy \cdot f_Y(y)\, e^{juy}\, dy$$

$$\frac{d^2}{du^2}\Phi_Y = \int_{-\infty}^{\infty} (jy)^2 f_Y(y)\, e^{juy}\, dy$$

$$\vdots$$

$$\frac{d^n}{du^n}\Phi_Y = \int_{-\infty}^{\infty} (jy)^n f_Y(y)\, e^{juy}\, dy$$

At the point $u = 0$:

$$\left.\frac{d^n}{du^n}\Phi_Y\right|_{u=0} = j^n \int_{-\infty}^{\infty} y^n f_Y(y)\, dy$$

since $e^{j0 \cdot y} = 1$ at $u = 0$.

$$= j^n \cdot (n^{th}\text{-order moment of } f_Y(y)).$$

For the above ~~case~~, with $n = 1$,

$$\cancel{\frac{d}{du}\Phi_Y(u) = \frac{d}{du}\left[\frac{\sin^3(uA/2)}{(uA/2)^3} \cdot e^{juA/2}\right]}$$

$$\cancel{=}$$

3 (d - cont)

For the above case with $n = 1$

$$\frac{d}{du}\Phi_Y = \frac{d}{du}\left[\operatorname{sinc}^3\left(\frac{uA}{2}\right).e^{juA/2}\right]$$

$$= 3\operatorname{sinc}^2\left(\frac{uA}{2}\right).\frac{d}{du}\left(\operatorname{sinc}\left(\frac{uA}{2}\right)\right).e^{juA/2}$$

$$+ \operatorname{sinc}^3\left(\frac{uA}{2}\right).\frac{jA}{2}.e^{juA/2}$$

Now at $u=0$, $\frac{d}{du}\left(\operatorname{sinc}\left(\frac{uA}{2}\right)\right) = 0$ since the sinc function has zero gradient at its centre point, and $\operatorname{sinc}\left(\frac{uA}{2}\right) = 1$ at this point.

$$\therefore \left.\frac{d}{du}\Phi_Y\right|_{u=0} = 0 + (1)^3.\frac{jA}{2}.1 = \frac{jA}{2}$$

Hence the first-order moment (or mean value) of $f_Y(y)$ is $\frac{A}{2}$, when we divide $\frac{d}{du}\Phi_Y$ by j.

This is what we would expect as the mean of X_1 is zero & the mean of X_2 is $\frac{A}{2}$, by inspection of the pdfs, f_1 & f_2, and $Y = X_1 + X_2$

4. (a) The matrix for a valid joint pdf should sum to unity.

The columns of $P(X_{n-1}, X_n)$ sum to $0.3, 0.4, 0.3$ & hence the matrix does sum to unity, and is valid.

The matrix is symmetric, so both the rows & columns sum to $[0.3 \quad 0.4 \quad 0.3]$ & hence this gives the probabilities for the 3 states, A, B and C, respectively.

$$\therefore \left.\begin{aligned} P(X_n = A) &= 0.3 \\ P(X_n = B) &= 0.4 \\ P(X_n = C) &= 0.3 \end{aligned}\right\} \text{ for any } n.$$

(b) The conditional table defines $P(X_n | X_{n-1})$

But $P(X_n, X_{n-1}) = P(X_n | X_{n-1}) . P(X_{n-1})$

$$\therefore P(X_n | X_{n-1} = A) = \frac{P(X_n, X_{n-1} = A)}{P(X_{n-1} = A)} = \begin{bmatrix} 0.2 \\ 0.05 \\ 0.05 \end{bmatrix} \cdot \frac{1}{0.3}$$

Similarly for $X_{n-1} = B$ & C:

$$\therefore P(X_n | X_{n-1}) = \begin{bmatrix} 0.667 & 0.125 & 0.167 \\ 0.167 & 0.75 & 0.167 \\ 0.167 & 0.125 & 0.667 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & \frac{1}{8} & \frac{1}{6} \\ \frac{1}{6} & \frac{3}{4} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{8} & \frac{2}{3} \end{bmatrix}$$

(fractions are more accurate)

4. (b) (cont.)

Mutual info: $I(X_n; X_{n-1}) = H(X_n) - H(X_n | X_{n-1})$

$$H(X_n) = -\sum_{i=1}^{3} p_i \log_2 p_i = -\left(2 . 0.3 \log_2 0.3 + 0.4 \log_2 0.4\right)$$

$$= 2 . 0.5211 + 0.5288 = \underline{\underline{1.5710}} \text{ bit/symbol}$$

$$H(X_n | X_{n-1} = A) = -\left(2 . \tfrac{1}{6} \log_2 \tfrac{1}{6} + \tfrac{2}{3} \log_2 \tfrac{2}{3}\right)$$

$$= 2 . 0.4308 + 0.3900 = 1.2516 \text{ bit/symbol}$$

$$H(X_n | X_{n-1} = B) = -\left(2 . \tfrac{1}{8} \log_2 \tfrac{1}{8} + \tfrac{3}{4} \log_2 \tfrac{3}{4}\right)$$

$$= 2 . 0.3750 + 0.3113 = 1.0613 \text{ bit/symbol}$$

$$H(X_n | X_{n-1} = C) = H(X_n | X_{n-1} = A) = 1.2516 \text{ bit/symbol}$$

$$\therefore H(X_n | X_{n-1}) = 2 . 0.3 . 1.2516 + 0.4 . 1.0613$$

$$= 2 . 0.3755 + 0.4245 = \underline{\underline{1.1755}} \text{ bit/sym}$$

$$\therefore I(X_n; X_{n-1}) = 1.5710 - 1.1755 = \underline{\underline{0.3955 \text{ bit/sym}}}$$

4. (c) For a block of 4 symbols:

Total entropy $= H(X_{n-3}) + H(X_{n-2}|X_{n-3})$
$+ H(X_{n-1}|X_{n-2}) + H(X_n|X_{n-1})$

$= H(X_n) + 3H(X_n|X_{n-1})$

$= 4H(X_n) - 3I(X_n;X_{n-1})$

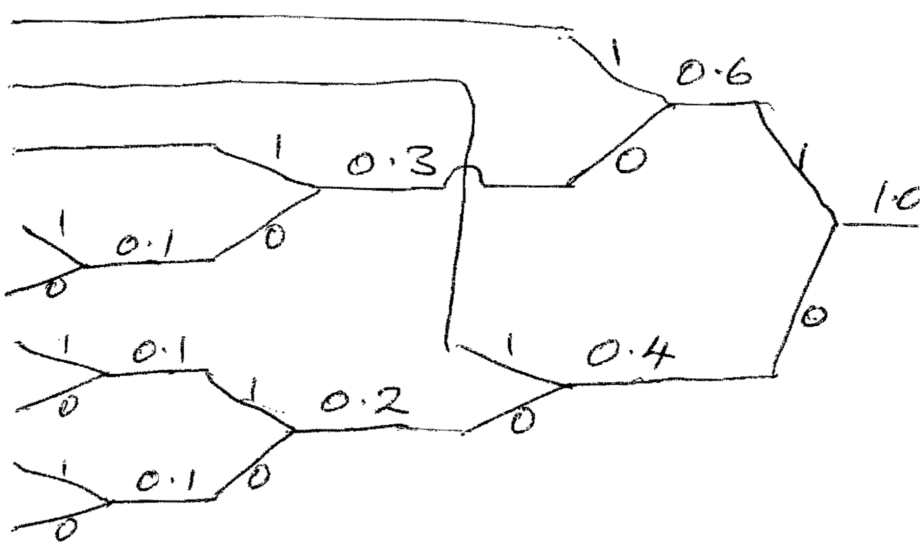
$= 4.1{\cdot}5710 - 3.0{\cdot}3955 = 6{\cdot}2840 - 1{\cdot}1865$

$= \underline{\underline{5{\cdot}0975}}$ bits/block.

This is the average no. of bits to encode each block, assuming an ideal encoder.

(d)

11	BB	0·3
01	AA	0·2
101	CC	0·2
1001	AB	0·05
1000	AC	0·05
0011	BA	0·05
0010	BC	0·05
0001	CA	0·05
0000	CB	0·05



Av. no of bits to code 2 symbols $= 2(0{\cdot}3 + 0{\cdot}2) + 3.0{\cdot}2$
$+ 4.6.0{\cdot}05$

$= 1{\cdot}0 + 0{\cdot}6 + 1{\cdot}2 = 2{\cdot}8$ bit/(2 sym)

$\therefore$ Av no of bits to code 4 symbols $= 2.2{\cdot}8 = \underline{\underline{5{\cdot}6 \text{ bits}}}$

Efficiency $= \dfrac{\text{Entropy}}{\text{Av no. of bits}} = \dfrac{5{\cdot}0975}{5{\cdot}6} = \underline{\underline{91{\cdot}03\%}}$

4. e) The main loss of efficiency in (d) is because we have not taken account of the mutual info between symbols 2 + 3 of the 4-bit block.
This would reduce the word length by approx 0.3955 bits to $5.6 - 0.3955 = 5.2045$ bits/block, which would increase the efficiency to $\frac{5.0975}{5.2045} = \underline{97.94\%}$

Arithmetic coding would probably be the easiest way to increase the efficiency in practise, since one could start each block with $P(X_n) = [0.3\ \ 0.4\ \ 0.3]^T$ as the prob distribution for coding the first symbol, and then use appropriate columns from $P(X_n | X_{n-1})$ for each subsequent symbol X_n, given knowledge of the previous symbol X_{n-1}.
This should get a result very close to the calculated entropy of 5.0975 bit/block = $\underline{1.2744}$ bit/sym.

If one used Arithmetic coding over much longer blocks, then one should be able to approach closely the conditional entropy, $H(X_n | X_{n-1}) = \underline{1.1755}$ bit/sym.