3F2 Systems and Control: 2013 Solutions

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1. (a) If (x_0, u_0) is an equilibrium pair, then by definition $f(x_0, u_0) = 0$. Suppose that x and u are perturbed from the equilibrium values, so that $x = x_0 + \delta x$ and $u = u_0 + \delta u$. Then $\dot{x} = \delta\dot{x}$. We thus have, by Taylor series expansion:

$$\begin{aligned} \delta\dot{x} &= f(x_0 + \delta x, u_0 + \delta u) & (1)\\ &= f(x_0, u_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,u_0)} \delta x + \left.\frac{\partial f}{\partial u}\right|_{(x_0,u_0)} \delta u + \text{ higher-order terms} & (2)\\ &\approx 0 + A\delta x + B\delta u & (3) \end{aligned}$$

where the higher-order terms have been neglected, the (i, j) term of the matrix A is $\partial f_i/\partial x_j$, and the (i, j) term of the matrix B is $\partial f_i/\partial u_j$, these partial derivatives being evaluated at $x = x_0, u = u_0$. Similar expansion of $y_0+\delta y = g(x_0+\delta x, u_0+\delta u)$ leads to $\delta y \approx C\delta x+D\delta u$, where the (i, j) term of the matrix C is $\partial g_i/\partial x_j$, and the (i, j) term of the matrix D is $\partial g_i/\partial u_j$, and noting that $y_0 = g(x_0, u_0)$.

(b) i. For an equilibrium we must have $\ddot{h} = 0$, and hence $g - (u_0/mh_0^2) = 0$, or $u_0 = mgh_0^2$, so

$$h_0 = \sqrt{\frac{u_0}{mg}} \tag{4}$$

ii. The ball height is governed by a second-order differential equation, so we expect to need 2 state variables (and this is also clear from what we are asked to show.) Since the given output equation is $\delta h = [1, 0]\delta x$, it is clear that the appropriate choice of state variables is $x_1 = h, x_2 = \dot{h}$. From this it is clear that

$$\delta\dot{x} = \begin{bmatrix} 0 & 1 \\ ? & ? \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ ? \end{bmatrix} \delta u \tag{5}$$

since $f_1(x, u) = x_2$. To find the remaining elements of A and B we have $f_2(x, u) = g - (u/mx_1^2)$. Hence

$$\begin{aligned} A_{2,1} &= \left.\frac{\partial f_2}{\partial x_1}\right|_{(h_0,u_0)} = \left.\frac{2u}{mx_1^3}\right|_{(h_0,u_0)} = \frac{2g}{h_0} & (6)\\ A_{2,2} &= \left.\frac{\partial f_2}{\partial x_2}\right|_{(h_0,u_0)} = 0 & (7)\\ B_{2,1} &= \left.\frac{\partial f_2}{\partial u}\right|_{(h_0,u_0)} = -\left.\frac{1}{mx_1^2}\right|_{(h_0,u_0)} = -\frac{1}{mh_0^2} & (8) \end{aligned}$$

Thus we have

$$\delta\dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{2g}{h_0} & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ -\frac{1}{mh_0^2} \end{bmatrix} \delta u \tag{9}$$

as required.

iii. The eigenvalues of the matrix A are the roots of $\det(sI - A) = s^2 - (2g/h_0) = 0$, namely $s = \pm\sqrt{2g/h_0}$. So both roots are real, with one being positive. The system is therefore $\boxed{\text{unstable}}$.

iv. The transfer function is given by the standard formula (easily derived if not remembered)

$$\begin{aligned} G(s) &= D + C(sI - A)^{-1}B & (10)\\ &= 0 + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ -\frac{2g}{h_0} & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -\frac{1}{mh_0^2} \end{bmatrix} & (11)\\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\begin{bmatrix} s & 1 \\ \frac{2g}{h_0} & s \end{bmatrix}}{s^2 - \frac{2g}{h_0}} \begin{bmatrix} 0 \\ -\frac{1}{mh_0^2} \end{bmatrix} & (12)\\ &= \frac{\begin{bmatrix} s & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{1}{mh_0^2} \end{bmatrix}}{s^2 - \frac{2g}{h_0}} & (13)\\ &= -\frac{1}{mh_0^2(s^2 - \frac{2g}{h_0})} & (14) \end{aligned}$$

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2. (a) For a (negative) feedback system, the return ratio is the open-loop transfer function obtained when a feedback loop is cut at some point, conventionally omitting the sign inversion. It is the product of all the transfer functions connected in series around the loop. (In a multivariable feedback system the return ratio is not unique, because transfer function matrices do not commute [$GK \neq KG$ etc] so the point at which the loop is broken needs to be specified.)

(b) If $L(s)$ is the return ratio, then the closed-loop poles (assuming a negative feedback convention) are the roots of

$$1 + L(s) = 0 \qquad \text{(or } \det(I + L(s)) = 0 \text{ in a multivariable feedback system).} \qquad (15)$$

Derivation of this (not asked for): Suppose a signal u is applied as the input to the 'cut' end of the feedback loop, and y is the signal obtained at the output. Then $\bar{y}(s) = -L(s)\bar{u}(s)$. When the feedback loop is closed we have $u \equiv y$, so $\bar{u}(s) = -L(s)\bar{u}(s)$, or $[1 + L(s)]\bar{u}(s) = 0$. For this to be true with non-trivial u we must have that $1 + L(s) = 0$.

(c) A root-locus diagram assumes that the return ratio has the form $L(s) = kG(s)$ for some constant real gain k and some transfer function $G(s)$. It traces the roots of

$$1 + L(s) = 1 + kG(s) = 0 \qquad (16)$$

as k varies. This can be re-written as

$$G(s) = -\frac{1}{k} \qquad (17)$$

If s_o is a point on the root locus then it must solve this equation, and so if $k > 0$ then

$$\arg G(s_o) = (2n+1)\pi \qquad \text{for any integer } n \qquad (18)$$

(or $\arg G(s_o) = 2n\pi$ if $k < 0$). This is called the *angle criterion.*

(Answers which emphasise the geometrical method of testing this condition for rational transfer functions can be accepted, providing that they give some explanation of the criterion.)

(d) i. The poles are at the roots of $s^2 + 25$, namely $\pm 5j$, and at the roots of $s^2 + 2s + 5 = (s+1)^2 + 4$, namely $-1 \pm 2j$. The zeros are at the roots of $s^2 + s + 5/4 = (s + 1/2)^2 + 1$, namely at $-\frac{1}{2} \pm j$. So the pole and zero locations are as shown in Fig.1.

ii. This can be checked in (at least) two ways:
1. By showing that the angle criterion cannot be satisfied on the imaginary axis (except at the imaginary poles, for $k = 0$):
Suppose that a point s_o on the root locus is on the imaginary axis, as shown in Fig.2, where it has been assumed that $|s_o| < 5$. To check the angle criterion we consider the arguments of all the complex numbers $s_o - p_i$, for $i = 1, 2, 3, 4$ (where p_i denotes the ith pole), and $s_o - z_i$ for $i = 1, 2$ (where z_i denotes the ith zero). We have $\arg(s_o - 5j) = -\pi/2$ and $\arg(s_o + 5j) =$

$+\pi/2$ so the net angle contribution from the imaginary poles is zero. Now $|\arg(s_o+1-2j)-\arg(s_o+1/2-j)|$ is the angle a shown in Fig.2, so this is the net angle contribution from the complex pole-zero pair in the upper half of the complex plane. It is clear that $0<a<\pi/2$. By similar reasoning it can be seen that the net angle contribution from the pole-zero pair in the lower half of the complex plane is also smaller than $\pi/2$. Hence the net contribution from all the poles and zeros is smaller than π, and hence the angle criterion cannot be satisfied at s_o.
If $|s_o|>5$, so that the trial point is on the imaginary axis above $5j$, then a similar argument holds. The difference now is that $\arg(s_o-5j)=+\pi/2$ so the net contribution from the two imaginary poles is now π. But the contribution from the remaining poles and zeros is again larger than 0 and smaller than π, so again the angle criterion cannot be satisfied.
2. By using the Routh-Hurwitz criterion — this is a bit risky, as it will involve 4th degree polynomials and it is not clear in advance whether a condition can be obtained easily. But it does work out ok in this case: If $L(s)=kn(s)/d(s)$ then $1+L(s)=0$ has the same roots as $d(s)+kn(s)=0$. We have

$$\begin{aligned} d(s)+kn(s) &= (s^2+25)(s^2+2s+5)+k(s^2+s+\frac{5}{4}) && (19)\\ &= (s^4+2s^3+(25+5)s^2+50s+125)+ks^2+ks+\frac{5k}{4} && (20)\\ &= s^4+2s^3+(30+k)s^2+(50+k)s+(125+\frac{5k}{4}) && (21)\\ &\equiv a_4s^4+a_3s^3+a_2s^2+a_1s+a_0 && (22)\end{aligned}$$

Now, from the Electrical and Information Data Book, the Routh-Hurwitz condition for all the roots to be in the left half-plane for $n=4$ is $a_i>0$ for all i and

$$a_1a_2a_3 > a_0a_3^2+a_4a_1^2 \qquad (23)$$

which in this case is

$$\begin{aligned} (50+k)(30+k)2 &> (125+\frac{5k}{4})4+(50+k)^2 && (24)\\ \Leftrightarrow 3000+160k+2k^2 &> 3000+105k+k^2 && (25)\\ \Leftrightarrow k^2+55k &> 0 && (26)\end{aligned}$$

which is clearly true for any $k>0$.

iii. Following the usual root-locus construction rules (for $k>0$):
1. No part of the real axis is on the root locus (always to the left of an even number of poles and zeros).
2. Two loci will terminate on the two zeros. $4-2=2$ loci will approach asymptotes as $k\to\infty$.
3. The asymptotes will intersect the real axis at the 'centre of gravity of poles and zeros':

$$\frac{\sum\text{poles}-\sum\text{zeros}}{\#\text{poles}-\#\text{zeros}}=\frac{-2+1}{4-2}=-\frac{1}{2} \qquad (27)$$

4. The angles of the asymptotes relative to the positive real axis will be

$$\frac{(2n+1)\pi}{\#\text{poles}-\#\text{zeros}}=\pm\frac{\pi}{2} \qquad (28)$$

This allows the root locus to be sketched, as shown in Fig.3. The possibility that the loci from the imaginary poles terminate on the zeros can be ruled out by reasoning similar to that used in part 2(d)ii above. Trying a point to the right of the asymptote but below $5j$ quickly shows that the angle criterion cannot be satisfied.

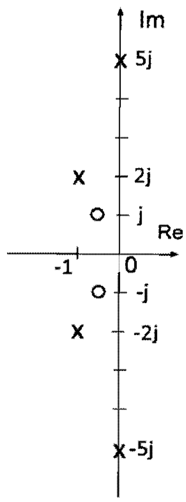

Figure 1:

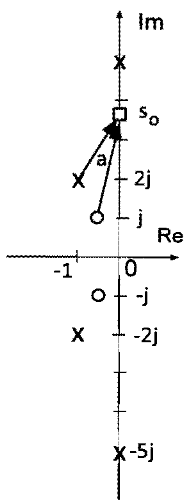

Figure 2:

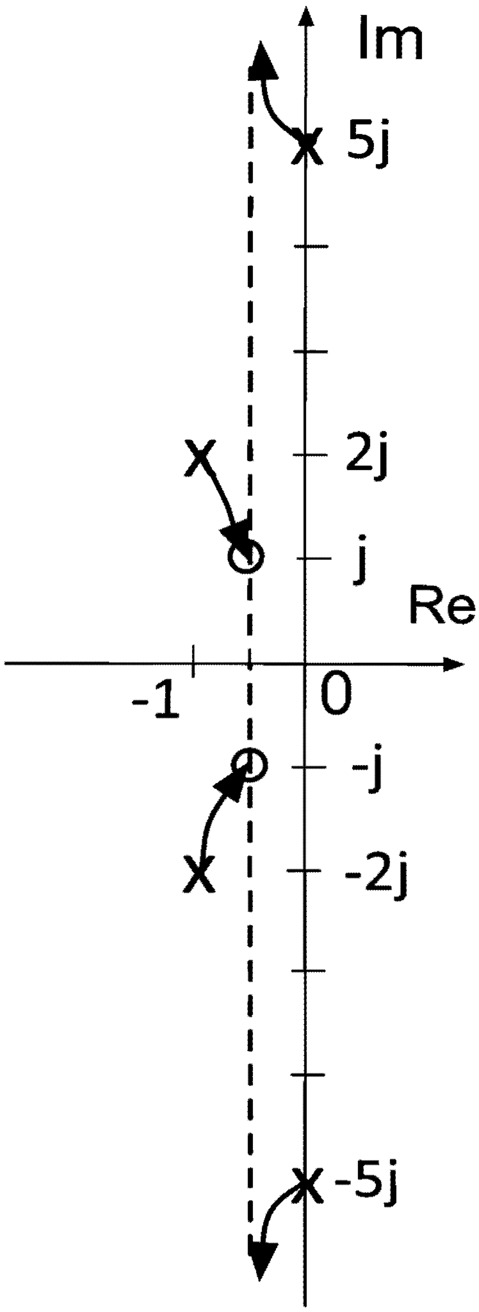

Figure 3:

3. (a) The equations of motion are two coupled 2nd-order ODE's, so we need a 4-dimensional state vector $x = [x_1, \ldots, x_4]^T$. A conventional choice of state variables is $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = z$, $x_4 = \dot{z}$. With this choice the equations in state-space form are:

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{z} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(m+M)g}{m\ell} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{Mg}{m} & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m\ell} \\ 0 \\ \frac{1}{m} \end{bmatrix} u \tag{29}$$

$$y = \begin{bmatrix} -a & 0 & 1 & 0 \end{bmatrix} x \tag{30}$$

(b) To test observability we need to check the rank of the observability matrix:

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} -a & 0 & 1 & 0 \\ 0 & -a & 0 & 1 \\ \frac{a(m+M)g}{m\ell} - \frac{Mg}{m} & 0 & 0 & 0 \\ 0 & \frac{a(m+M)g}{m\ell} - \frac{Mg}{m} & 0 & 0 \end{bmatrix} \tag{31}$$

(where the efficient way of computing CA^k is as $CA^{k-1} \times A$). The first two rows are clearly linearly independent of each other. Rows 3 and 4 are clearly independent of each other, and of rows 1 and 2, unless

$$\frac{a(m+M)g}{m\ell} - \frac{Mg}{m} = 0 \tag{32}$$

which is the same as

$$a = \frac{M\ell}{m+M} \tag{33}$$

Thus the observability matrix has full rank, and the system is observable, unless this condition holds.
Alternatives: 1. The rank can also be checked by evaluating the determinant of the observability matrix. In general this is to be avoided for a 4×4 matrix, but in this case there are so many zero entries that it is quite easy to do it by hand.
2. If the given value of a is substituted into the observability matrix then the 3rd and 4th rows contain only zeros and the loss of rank becomes obvious.

(c) $a = M\ell/(m + M)$ corresponds to having the camera monitoring the centre of mass of the cart-bob system. When no external force is applied to the cart, this point has no acceleration (remains stationary if it starts stationary). It is therefore not surprising that the system is unobservable if only this point is monitored.
Examiner's note: Surprisingly few people made the connection with the centre of mass.

(d) 'Closeness to unobservability' can be quantified by considering the 'energy' in the output signal $\int_0^\infty y(t)^T y(t) dt$ when the system is released from an initial condition $x(0)$ and no input is applied to the system.

$$\int_0^\infty y(t)^T y(t) dt = \int_0^\infty [Ce^{At}x(0)]^T [Ce^{At}x(0)] dt \tag{34}$$

$$= \int_0^\infty x(0)^T [e^{A^T t} C^T C e^{At}] x(0) dt \tag{35}$$

$$= x(0)^T \left[\int_0^\infty e^{A^T t} C^T C e^{At} dt \right] x(0) \tag{36}$$

$$= x(0)^T W_o x(0) \tag{37}$$

where W_o is the observability Gramian. (This development is correct if the system is stable. Otherwise the integral will not converge and a finite upper limit of time should be used instead.)
It can be shown that W_o is singular if and only if the system is unobservable, in which case there will be some non-zero $x(0)$ for which $x(0)^T W_o x(0) = 0$. 'Closeness to unobservability' can be measured by the closeness of W_o to singularity. One standard way of doing this is to see how close the smallest singular value of W_o is to zero.

A practical consequence of using a value of a close to that which corresponds to unobservability is that some components of x will give rise to very little 'energy' in the measured output, and this will result in poor signal-to-noise ratio in the measurement — since some noise is inevitable. The result will be poor estimation of the state.

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4. (a) A linear system is controllable if its state can be transferred from any initial state x_i to any final state x_f in a finite time τ by the application of some control $u(t), 0 \le t \le \tau$. (Variations of this definition are acceptable.)

(b) A linear system defined by the equations $\dot{x} = Ax + Bu$, with $x \in \mathbf{R}^n$, is controllable if and only if the controllability matrix $[B, AB, \ldots, A^{n-1}B]$ has full rank.

(c) The combined system has governing equation $\dot{x} = \tilde{A}x + \tilde{B}u$, where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbf{R}^{2n}, \quad \tilde{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ B \end{bmatrix} \tag{38}$$

so that the controllability matrix is

$$\begin{bmatrix} B & AB & \cdots & A^{2n-1}B \\ B & AB & \cdots & A^{2n-1}B \end{bmatrix} \tag{39}$$

Clearly the rows in the bottom half of this matrix are not linearly independent of those in the top half, so this matrix does not have full rank, and hence the combined system is unobservable.

(d) We now have $\dot{x}_2 = Ax_2 + BB^Tx_1$, so the combined system has governing equation $\dot{x} = \hat{A}x + \hat{B}u$, where

$$\hat{A} = \begin{bmatrix} A & 0 \\ BB^T & A \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} \tag{40}$$

and hence the controllability matrix is now

$$\begin{bmatrix} B & AB & \cdots \\ 0 & BB^TB & \cdots \end{bmatrix} \tag{41}$$

Clearly this can have full rank: for example consider any 1-state, 1-input system with $B = 1$. Then the controllability matrix of the connected system is

$$\begin{bmatrix} 1 & A \\ 0 & 1 \end{bmatrix} \tag{42}$$

which has full rank. Therefore identical systems connected in series in this way can be controllable.
Examiner's note: Sections (c) and (d) revealed a severe lack of familiarity with manipulating matrices among many candidates. Some did not realise that $[x_1^T, x_2^T]^T$ *was just the same as* $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, *but had* $\begin{bmatrix} x_1^T \\ x_2^T \end{bmatrix}$ *instead (which makes no sense, but that's another story). In part (d) many matrix expressions were written down which could not work dimensionally.*

(e) The series connection implies that $u_2 = y_1$ and $u_3 = y_2$, so we have $\dot{x}_2 = kx_1$ and $\dot{x}_3 = kx_2$. Thus the complete (open-loop) system has the equation

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ k & 0 & 0 \\ 0 & k & 0 \end{bmatrix} x + \begin{bmatrix} \sqrt{k} \\ 0 \\ 0 \end{bmatrix} u_1 \tag{43}$$

Let the state-feedback be $u_1 = -k_1x_1 - k_2x_2 - k_3x_3 = -[k_1,\ k_2,\ k_3]x$, then the closed-loop equation is

$$\dot{x} = \left(\begin{bmatrix} 0 & 0 & 0 \\ k & 0 & 0 \\ 0 & k & 0 \end{bmatrix} - \begin{bmatrix} \sqrt{k} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \right) x \tag{44}$$

$$= \begin{bmatrix} -k_1\sqrt{k} & -k_2\sqrt{k} & -k_3\sqrt{k} \\ k & 0 & 0 \\ 0 & k & 0 \end{bmatrix} x \tag{45}$$

To get the closed-loop poles we need the eigenvalues, and hence the characteristic polynomial, of this matrix:

$$
\begin{aligned}
\det(\lambda I - A_c) &= \begin{vmatrix} \lambda + k_1\sqrt{k} & k_2\sqrt{k} & k_3\sqrt{k} \\ -k & \lambda & 0 \\ 0 & -k & \lambda \end{vmatrix} && (46)\\
&= \lambda \begin{vmatrix} \lambda + k_1\sqrt{k} & k_2\sqrt{k} \\ -k & \lambda \end{vmatrix} + k \begin{vmatrix} \lambda + k_1\sqrt{k} & k_3\sqrt{k} \\ -k & 0 \end{vmatrix} && (47)\\
&= \lambda[(\lambda + k_1\sqrt{k})\lambda + kk_2\sqrt{k}] + k[kk_3\sqrt{k}] && (48)\\
&= \lambda^3 + k_1\sqrt{k}\lambda^2 + kk_2\sqrt{k}\lambda + k^2k_3\sqrt{k} && (49)
\end{aligned}
$$

But this should be the same as $(\lambda+1)^3 = \lambda^3 + 3\lambda^2 + 3\lambda + 1$. So comparing coefficients, we need

$$k_1 = 3/\sqrt{k}, \quad k_2 = 3/k\sqrt{k}, \quad k_3 = 1/k^2\sqrt{k} \tag{50}$$