AØ   Engineering tripos
module 3F4 - Data Transmission   2013.

1.) a) Equalisers are designed to minimize the inter symbol interference (ISI) present at the slicer in the receiver of a baseband digital communication system. Their purpose is to compensate for unknown or varying channels while not increasing the noise too much in order to improve the bit error rate (BER) at the receiver output.

Zero forcing equalizer

Is a linear sampled data (digital) filter which attempts to convert the sampled pulse response $P(z)$ at the equalizer input into a discrete delta function at the equalizer output, i.e.,...1,0,0,.... To achieve this the equalizer response is given by $H_E(z)=1/P(z)$. If $P(z)$ is FIR, then $H_E(z)$ will be IIR, which can cause stability problems. Consequently, $H_E(z)$ is often approximated by an FIR filter which may be designed to force a finite number of zeros around the wanted pulse. Zero forcing equalizers can suffer from significant noise magnification if $P(z)$ has nulls in its spectrum $P(e^{j\omega T})$.

Decision feedback equalizer

Essentially a modified IIR equalizer where the slicer is moved inside the feedback filter. Thus ISI in the channel response is cancelled by feeding back delayed decisions from the slicer with appropriate weightings. This is a non-linear filter and its main advantage is the elimination of noise on the fed back decisions. Consequently there is no noise magnification and an improvement in the BER compared with a conventional IIR based equalizer. However at high noise levels we can no longer assume that the fed back decisions are correct and so ISI is no longer cancelled correctly, leading to more decision errors. Consequently an error burst occurs which will last until the correct data is re-established in the feedback registers. This degrades performance. but it is still usually better than a zero forcing equalizer. 30% [6]

A1

(b) The pulse response is $p_0 = 1$, $p_1 = 0.2$, $p_2 = -0.1$
and the ZF constraint is $y_0 = 1$, $y_1 = 0$, $y_2 = 0$

In general,

$$y_n = p_n b_0 + p_{n-1} b_1 + p_{n-2} b_2 + \cdots p_{n-q} b_q$$

where $b_0 \cdots b_q$ are the filter coefficients.
The FIR filter has $q+1$ coefficients, i.e.,

$$q + 1 = 3 \quad \therefore \quad q = 2.$$

$$y_n = p_n b_0 + p_{n-1} b_1 + p_{n-2} b_2$$ 1

$n=0$ $\quad y_0 = p_0 b_0$ ①

$n=1$ $\quad y_1 = p_1 b_0 + p_0 b_1$ ②

$n=2$ $\quad y_2 = p_2 b_0 + p_1 b_1 + p_0 b_2$ ③

From ① $\quad 1 = 1 \times b_0 \quad \therefore \quad \underline{b_0 = 1}$ 1

From ② $\quad 0 = 0.2 \times 1 + 1 \times b_1 \quad \therefore \quad \underline{b_1 = -0.2}$ 1

From ③ $\quad 0 = (-0.1 \times 1) + (0.2 \times -0.2) + 1 \times b_2$

$\therefore \quad \underline{\underline{b_2 = 0.14}}$ 1 2016

[4]

(c)(i) Need to calculate the values of the residuals at the equaliser output.

$n = 3$ $\quad y_3 = p_2 b_1 + p_1 b_2$

$y_3 = (-0.1 \times -0.2) + (0.2 \times 0.14)$

$\underline{\underline{y_3 = 0.048}}$

$n = 4$ $\quad y_4 = p_2 b_2$

$y_4 = -0.1 \times 0.14$

$\underline{\underline{y_4 = -0.014}}$

For a polar binary scheme the worst-case '1' eye opening is $1 - 0.048 - 0.014 = 0.938$

$\uparrow$ one other '0' contributing $\quad$ $\uparrow$ one other '1' contributing

So total eye opening, $h = 2 \times 0.938$

$h = 1.876$ 1

Now need to find the noise variance,

$\sigma_w = \sigma_v \sqrt{b_0^2 + b_1^2 + b_2^2}$

$\sigma_w = 0.3 \sqrt{1^2 + (-0.2)^2 + (0.14)^2}$

$\sigma_w = 0.3 \times 1.0294$

$\sigma_w = 0.3088\ V$ 1

So, worst case equalised error rate (BER) is

$$P_e = Q\left(\frac{h}{2\sigma_w}\right) = Q\left(\frac{1.876}{2 \times 0.3088}\right)$$

$$P_e = Q(3.038) = 1.19 \times 10^{-3}$$ 1

Without equaliser,

minimum '1' eye opening $= 1 - 0.2 - 0.1$

1 other '0' contributing

1 other '1' contributing

$= 0.7$

minimum total eye opening, $h = 2 \times 0.7$

$h = 1.4$ 1

$$\therefore P_e = Q\left(\frac{1.4}{2 \times 0.3}\right) = Q(2.33) = 9.93 \times 10^{-3}$$ 1

50%

(8)

A3

b) (ii) ~~as the data are equiprobable~~ There are 4 possible eye opening values for the equalised system, specifically

$h_1 = (1 - 0.048 - 0.014) \times 2 = 1.876$
$h_2 = (1 - 0.048 + 0.014) \times 2 = 1.932$
$h_3 = (1 + 0.048 - 0.014) \times 2 = 2.068$
$h_4 = (1 + 0.048 + 0.014) \times 2 = 2.124$

Assuming equiprobable data the expected case BER performance is,

$$P_{exp} = 0.25\, Q\left(\frac{1.876}{0.6176}\right) + 0.25\left(\frac{1.932}{0.6176}\right) + 0.25\left(\frac{2.068}{0.6176}\right) + 0.25\left(\frac{2.124}{0.6176}\right)$$

$$P_{exp} = 0.25\left[Q(3.038) + Q(3.128) + Q(3.348) + (3.44)\right]$$ 1

$$P_{exp} = 0.25\left[1.19\times10^{-3} + 8.82\times10^{-4} + 6.7\times10^{-4} + 2.91\times10^{-4}\right]$$

$$= \underline{7.58\times10^{-4}}$$ 1

(2)

(c) DFE will remove all ISI with no noise enhancement.
Effect of error propagation is ignored. 1
$\therefore$ eye opening, $h = 2$ 1

$$BER = Q\left(\frac{2}{2\times0.3}\right) = Q(3.33) = \underline{4.35\times10^{-4}}$$

1

(3)

[10]

A1

2) a) This is the structure of each codeword for code C.

$$
\begin{array}{l|cccccccccccccccc}
 & d & d & d & p & d & d & d & p & d & d & d & p & p & p & p & p \\
\text{code bit} & 15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\
 & [\,d_8 & d_7 & d_6 & p_6 & d_5 & d_4 & d_3 & p_5 & d_2 & d_1 & d_0 & p_4 & p_3 & p_2 & p_1 & p_0\,]
\end{array}
$$

p_6: 15 → 13, $d_8 d_7 d_6$

p_5: 11 − 9, $d_5 d_4 d_3$

p_4: 7 − 5, $d_2 d_1 d_0$

p_3: 15, 11, 7, $d_8 d_5 d_2$

p_2: 14, 10, 6, $d_7 d_4 d_1$

p_1: 13, 9, 5, $d_6 d_3 d_0$

p_0: 12, 8, 4, $p_6 p_5 p_4$

Remember $s = cH^T$ so parity check

H has dimensions $(n-k) \times n = (16-9) \times 16 = 7 \times 16$

$$
H = \begin{array}{c}
 \\ p_6 \\ p_5 \\ p_4 \\ p_3 \\ p_2 \\ p_1 \\ p_0
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccc} 15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{array} \\
\left[\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

4

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A2

We can now write down the Generator matrix G, where $c = dG$.
G has dimension $(k \times n)$, ie (9×16)

data word $[d_8\ d_7\ d_6\ d_5\ d_4\ d_3\ d_2\ d_1\ d_0]$

$$G = \begin{array}{cccccccccccccccc}
15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\
\end{array}$$
$$\left[\begin{array}{cccccccccccccccc}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1
\end{array}\right]$$

codebit $\emptyset$ ($c_\emptyset$)

$c_\emptyset$ is a parity bit ($p_\emptyset$) given by codebits c_{12}, c_8 and c_4, ie, p_6, p_5, and p_4

p_6 is given by d_8, d_7, d_6
p_5 is given by d_5, d_4, d_3
p_4 is given by $d_2, d_1, d_\emptyset$

ie, to calc p_0 we need to do an even parity check on all the data bits in a data word ($d_8 \ldots d_\emptyset$), hence the all 1's final column of G.

4

[8]

A3

b) Even parity check on codebits 3–1 is given by,
c_3 (p_3) which is even parity check of d_8, d_5, d_2
c_2 (p_2) " " " " " " d_7, d_4, d_1
c_1 (p_1) " " " " " " d_6, d_3, d_0.

So in this case c_0 is given by an even parity check of $d_0 \dots d_8$, i.e., the same as that given by an even parity check of code bits c_{12}, c_8 and c_4, (i.e., p_6, p_5, p_4). [2]

c) The minimum distance of a code is equivalent to the minimum number of columns of H that sum to 0. For example –
By inspecting H it can be seen that summing columns 15, 14, 3 and 2 yields 0, i.e.,

$$\begin{array}{ccccccccc} c_{15} & & c_{14} & & c_3 & & c_2 & & \\ \begin{bmatrix}1\\0\\0\\1\\0\\0\\0\end{bmatrix} & \oplus & \begin{bmatrix}1\\0\\0\\0\\1\\0\\0\end{bmatrix} & \oplus & \begin{bmatrix}0\\0\\0\\1\\0\\0\\0\end{bmatrix} & \oplus & \begin{bmatrix}0\\0\\0\\0\\1\\0\\0\end{bmatrix} & = & \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\end{bmatrix}\end{array}$$

Thus ~~the~~ $d_{min} = 4$.

[3]

A4

d) Syndrome detection.

Error syndrome s of a received codeword c_r,

$$s = c_r H^T$$ 1

If c_r is corrupted by the addition of an error vector e,

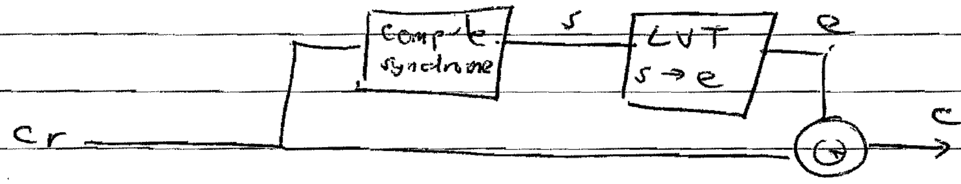
$$c_r = c + e$$

and

$$\begin{aligned} s &= (c+e) H^T \\ &= cH^T + eH^T \\ &= 0 + eH^T \end{aligned}$$ 1

See syndrome depends only on the error.

For code C, each s corresponds to a unique error pattern hence decoding takes the form,



1

[3]

A5

e) We can represent code C as follows,

$$\left[\begin{array}{ccc|c} d_8 & d_7 & d_6 & p_6 \\ d_5 & d_4 & d_3 & p_5 \\ d_2 & d_1 & d_0 & p_4 \\ \hline p_3 & p_2 & p_1 & p_0 \end{array}\right] = \left[\begin{array}{ccc|c} c_{15} & c_{14} & c_{13} & c_{12} \\ c_{11} & c_{10} & c_9 & c_8 \\ c_7 & c_6 & c_5 & c_4 \\ \hline c_3 & c_2 & c_1 & c_0 \end{array}\right]$$

2

A single bit error will cause a parity check error in the associated row and column thus identifying the bit in error. This bit must then be complemented to correct the error.

This approach is much less complicated than the syndrome based decoder.

[4]

Q3. (a) $p(t) = e^{j\phi_0} \sum_k b_k \, g(t - kT_b)$

A real modulated signal is generated from a phasor waveform $p(t)$ by

$$s(t) = \text{Re}\left[p(t) \, e^{j\omega_c t} \right]$$

where ω_c is the carrier freq in rad/s. Hence

$$s(t) = \text{Re}\left[e^{j\phi_0} \sum_k b_k \, g(t - kT_b) \, e^{j\omega_c t} \right]$$

$$= \left[\sum_k b_k \, g(t - kT_b) \right] . \cos(\omega_c t + \phi_0)$$

$g(t)$ is positive, & $b_k = \pm 1$, & on the k^{th} bit period the waveform is $\pm a_0 \cos(\omega_c t + \phi_0)$.

$\therefore$ The two carrier phases are ϕ_0 and $\phi_0 + \pi$, depending on the polarity of b_k.

(b) $r(t) = p(t) + p_N(t)$

$$= p(t) + \left(n_1(t) + j\, n_2(t) \right) e^{j\phi_0}$$

$$= e^{j\phi_0} \left[\sum_k b_k \, g(t - kT_b) + n_1(t) + j n_2(t) \right]$$

$$y(k) = G \int_{kT_b}^{(k+1)T_b} \text{Re}\left[r(t) \, g(t - kT_b) \, e^{-j\phi_0} \right] dt$$

$$= G \int_{kT_b}^{(k+1)T_b} \text{Re}\left[\sum_k b_k \, g(t - kT_b) + n_1(t) + j\, n_2(t) \right] g(t - kT_b) \, dt$$

Q3. (b cont.)

$$= G \int_{kT_b}^{(k+1)T_b} Re\left[b_k a_o + n_1(t) + j n_2(t) \right] . a_o \, dt$$

$$= G \int_{kT_b}^{(k+1)T_b} \left[b_k a_o^2 + a_o n_1(t) \right] dt$$

$$\therefore y(k) = G a_o \left[b_k a_o T_b + \int_{kT_b}^{(k+1)T_b} n_1(t) \, dt \right]$$

This only depends on $n_1(t)$, as $n_2(t)$ disappears in the $Re[.]$ operation on $(n_1 + j n_2)$.

(c) ~~This~~ The signal component is $G a_o^2 b_k T_b$ & the noise component is $G a_o \int_{kT_b}^{(k+1)T_b} n_1(t) \, dt$.

The two signal voltages will be $\pm G a_o^2 T_b$.

Using freq domain analysis, the noise voltage is

$$G \int_{kT_b}^{(k+1)T_b} g(t - kT_b) \, n_1(t) \, dt$$

$$= G \int_{-\infty}^{\infty} n_1(t) \, h((k+1)T_b - t) \, dt \quad \text{where } h(t) = g(T_b - t)$$

This is a convolution with a filter with impulse resp. $h(t)$ so we use Parseval's theorem to analyse this in freq domain.

Q3 (c cont).

Mean noise power from filter $= G^2 \int_{-\infty}^{\infty} |N_1(\omega)\, H(\omega)|^2 \, d\omega$

$$= G^2 \int_{-\infty}^{\infty} |N_1(\omega)|^2 . |H(\omega)|^2 \, d\omega$$

But $|N_1(\omega)|^2 = \frac{N_0}{2}$ watt/Hz $= \frac{N_0}{4\pi}$ watt/(rad/s),
since the noise power in p_N is split equally between n_1 & n_2.

$\therefore$ Mean noise power from filter $= \frac{G^2 N_0}{4\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 \, d\omega$

$= \frac{G^2 N_0}{2} \int_{-\infty}^{\infty} |h(t)|^2 \, dt$ by Parseval's theorem

$$= \frac{G^2 N_0}{2} . a_0^2 T_b = \frac{G^2 N_0}{2} . E_b$$

since $E_b = a_0^2 T_b$, the energy per bit in $p(t)$.

$\therefore$ RMS noise voltage in $y(k)$, $= \sqrt{\frac{G^2 N_0 E_b}{2}}$

$\therefore$ Signal/Noise voltage ratio $= G a_0^2 T_b \Big/ \sqrt{\frac{G^2 N_0 E_b}{2}}$

$$= G E_b \Big/ G \sqrt{\frac{N_0 E_b}{2}} = \underline{\underline{\sqrt{\frac{2 E_b}{N_0}}}}$$

Q3 (d) When ~~Additive white~~ Gaussian noise is added to a signal of voltage $\pm V$ at a detector, the probability of error, when the detection threshold is zero, is given by

$$\text{Prob of error} = Q\left(\frac{V}{\sigma}\right) \quad \text{where } Q(x) = \int_x^{\infty} f(u)\,du$$

& $f(u)$ is a unit variance, zero mean, Gaussian pdf.

Hence in this case

$$\text{Prob of error} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

since $\frac{V}{\sigma}$ is the voltage SNR.

There are standard approximation formulae or tables to allow $Q(\cdot)$ to be easily calculated.

e) The probability of error for QPSK is also $Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$, the same as for BPSK. This is because half of the signal power appears on each ~~carrier~~ of the two quadrature subcarriers of QPSK, but the symbol period is also doubled, so the energy per bit on each subcarrier is the same as for the overall QPSK signal. The noise is independent of modulation.

∴ E_b/N_0 remains the same for BPSK and QPSK.

∴ QPSK can be demodulated as 2 separate BPSK signals.

Q4. (a) The Power Spectral Density $P(\omega)$ is the Fourier Transform of the autocorrelation function, $r_{bp}(\tau)$.

The signal spectrum $S(\omega)$ is the spectrum $P(\omega)$, shifted up by ω_c & down by $-\omega_c$, where ω_c is the carrier frequency & the shifted down spectrum is negated in freq & conjugated.

Hence $S(\omega) = \frac{1}{2}\left[P(\omega-\omega_c) + P^*(-(\omega+\omega_c))\right]$

(b) The autocorrelation function of a modulated multi-level signal is derived from the discrete ACF of the symbols: $R(L) = E(S_k \cdot S_{k-L})$

If S_k and S_{k-L} are uncorrelated and zero-mean when $L \neq 0$, then $E(S_k \cdot S_{k-L}) = 0$ when $L \neq 0$.

When $L=0$, $E(S_k \cdot S_{k-L}) = E(S_k^2) = E_s$, the energy per symbol

Hence $R(L) = \begin{cases} E_s & \text{for } L = 0 \\ 0 & \text{for } L \neq 0 \end{cases}$

Q4 (b - cont.)

The autocorrelation func. of the stream of data impulses $b(t) = \sum_k s_k \delta(t - kT_s)$ is

$$r_{bb}(\tau) = \frac{1}{T_s} \sum_L R(L)\, \delta(t - kT_s) = \frac{1}{T_s} \cdot E_s\, \delta(\tau)$$

The Power Spectrum of $b(t)$ is then

$$\therefore\ |B(\omega)|^2 = \int_{-\infty}^{\infty} r_{bb}(\tau)\, e^{-j\omega\tau}\, d\tau = \frac{E_s}{T_s}.$$

We want the power spectrum of $p(t) = \sum_k s_k\, g(t - kT_s)$

where $g(t)$ is a rectangular pulse of duration T_s.

$$\therefore\ |P(\omega)|^2 = |B(\omega)|^2\, |G(\omega)|^2 = \frac{E_s}{T_s} \cdot a_0^2 T_s^2 \operatorname{sinc}^2\left(\frac{\omega T_s}{2}\right)$$

$$= E_s \cdot a_0^2 T_s \operatorname{sinc}^2\left(\frac{\omega T_s}{2}\right)$$

Hence the bandwidth of the sinc^2 function depends only on $\frac{1}{T_s}$ (the symbol rate) and not on the number of modulation levels.

(c) In a M^2-QAM system there are 2 quadrature subcarriers, each conveying $\log_2 M$ bits per transmitted symbol, using M-level amplitude modulation (with zero mean).

Hence $T_s = 2 \cdot \log_2 M \cdot T_b$, where T_b is the bit period of the data stream.

Q4 (c - cont.)

Given the $\mathrm{sinc}^2\left(\frac{\omega T_s}{2}\right)$ term in the expression for $|P(\omega)|^2$ in part (b), the main lobe will extend from

$$\frac{\omega T_s}{2} = -\pi \quad \text{to} \quad \frac{\omega T_s}{2} = +\pi .$$

ie from $f = -\frac{2\pi}{T_s} \cdot \frac{1}{2\pi} = -\frac{1}{T_s}$ to $f = +\frac{1}{T_s}$ Hz.

$\therefore$ Width of main lobe $= \frac{2}{T_s}$ Hz $= \underline{\underline{\frac{1}{(\log_2 M) \cdot T_b}}}$.

(d) As M increases, then the bandwidth reduces in proportion to $\frac{1}{\log_2 M}$. This reduces quite quickly at first ~~at~~ for $M = 2, 4, 8$ but rapidly becomes slower as M increases.

On the other hand the resilience to noise reduces quickly as M increases since the distance between constellation states is proportional to $\sqrt{E_s}/M$.

Hence small M (eg ~~2~~ $M=2$ for QPSK) are used where bandwidth can be relatively wide, but ~~t~~ resilience to noise is very important (eg Digital Audio Broadcasting to vehicles), whereas larger M (eg $M=8$ giving 64-QAM) are used where bandwidth must be ~~[illegible]~~ minimised & where noise can be reduced by high gain directional antennae (eg. Digital TV to homes.).