1. (a) A Symmetric matrix A is positive definite if

$$\underline{x}^t A \underline{x} > 0 \quad \text{for all non-zero } \underline{x}$$

[10%]

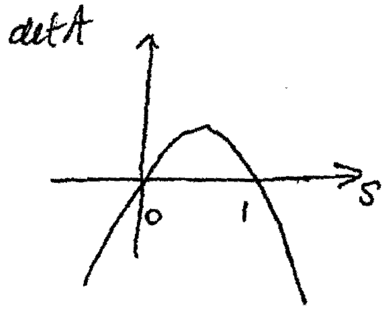
(b) A positive definite if & only if determinant of all principal minors is positive

For $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -s \\ -1 & -s & 1 \end{bmatrix}$ Minors (i) $2 > 0$ ✓

(ii) $\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1 > 0$ ✓

(iii) $\det A = 2(1-s^2) - 1(1-s) - 1(-s+1)$

$$\Rightarrow \det A = 2 - 2s^2 - 2 + 2s = 2s - 2s^2 = 2s(1-s)$$



$\therefore \det A > 0$ for $\underline{0 < s < 1}$

For this range A +ve definite

[15%]

(c) $s = \tfrac{1}{2}$

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -\tfrac{1}{2} \\ -1 & -\tfrac{1}{2} & 1 \end{bmatrix} = \begin{array}{c|c} & \begin{matrix} \sqrt{2} & \tfrac{1}{\sqrt{2}} & -\tfrac{1}{\sqrt{2}} \end{matrix} \\ \begin{matrix} \sqrt{2} \\ \tfrac{1}{\sqrt{2}} \\ -\tfrac{1}{\sqrt{2}} \end{matrix} & \begin{bmatrix} 2 & 1 & -1 \\ 1 & \tfrac{1}{2} & -\tfrac{1}{2} \\ -1 & -\tfrac{1}{2} & \tfrac{1}{2} \end{bmatrix} \end{array} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \tfrac{1}{2} & 0 \\ 0 & 0 & \tfrac{1}{2} \end{bmatrix}$$

$$= \begin{array}{c|c} & \begin{matrix} 0 & \tfrac{1}{\sqrt{2}} & 0 \end{matrix} \\ \begin{matrix} 0 \\ \tfrac{1}{\sqrt{2}} \\ 0 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & \tfrac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} + \begin{array}{c|c} & \begin{matrix} 0 & 0 & \tfrac{1}{\sqrt{2}} \end{matrix} \\ \begin{matrix} 0 \\ 0 \\ \tfrac{1}{\sqrt{2}} \end{matrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tfrac{1}{2} \end{bmatrix} \end{array}$$

$$\Rightarrow A = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \tfrac{1}{\sqrt{2}} & \tfrac{1}{\sqrt{2}} & 0 \\ -\tfrac{1}{\sqrt{2}} & 0 & \tfrac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \tfrac{1}{\sqrt{2}} & -\tfrac{1}{\sqrt{2}} \\ 0 & \tfrac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \tfrac{1}{\sqrt{2}} \end{bmatrix}$$

[25%]

(d) A symmetric $\Rightarrow$ e-values real & e-vectors can be chosen orthogonal & unit.

$$\Rightarrow \quad AU = U\Lambda \qquad \text{where} \quad U = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \underline{u}_1 & \underline{u}_2 & \cdots & \underline{u}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$\& \quad \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \text{ diagonal matrix} \qquad \text{unit e-vectors}$$

For any $\underline{x}$, we can write $\underline{x} = \alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 + \dots + \alpha_n \underline{u}_n$

$$\Rightarrow \underline{x}^t A \underline{x} = (\alpha_1 \underline{u}_1^t + \dots + \alpha_n \underline{u}_n^t)(\lambda_1 \alpha_1 \underline{u}_1 + \dots + \lambda_n \alpha_n \underline{u}_n)$$

$$= \lambda_1 \alpha_1^2 + \lambda_2 \alpha_2^2 + \dots + \lambda_n \alpha_n^2$$ since $\underline{u}$'s e-vectors & mutually orthogonal

All λ's +ve $\Rightarrow$ $\underline{x}^t A \underline{x} > 0$ for all $\underline{x}$ provided not all α's $= 0$
$\Rightarrow$ A positive definite

AND if $\lambda_1 < 0$ say take $\underline{x} = \underline{u}_1$

$\Rightarrow \underline{x}^t A \underline{x} = \lambda_1 < 0 \quad \Rightarrow$ A not positive definite

$\therefore$ <u>A positive definite if & only if</u> all λ's > 0 [25%]

(e) $A = U \Lambda U^t$ since $U^t U = I$

Define $$\tilde{\Lambda} = \begin{bmatrix} -\sqrt{\lambda_1} & & & \\ & -\sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & -\sqrt{\lambda_n} \end{bmatrix}$$ So $\tilde{\Lambda}^2 = \Lambda$

$$\therefore A = U \tilde{\Lambda}^2 U^t = U \tilde{\Lambda} U^t U \tilde{\Lambda} U^t$$

$$= B^2 \quad \text{where} \quad B = U \tilde{\Lambda} U^t. \quad (\Rightarrow B \text{ symmetric})$$

e-vectors of B are $\underline{u}_1, \dots, \underline{u}_n$; e-values $-\sqrt{\lambda_1}, \dots, -\sqrt{\lambda_n}$ which are -ve
$\therefore$ B negative definite. [25%]

Q2

(a) The task is to <u>maximise</u> the rate of heat transfer, which is proportional to the exposed area

$$T = C_1\left(\pi HD + \pi\frac{D^2}{4}\right) = C_2\left(HD + \frac{D^2}{4}\right)$$

Thus $\quad$ <u>Minimize</u> $f(D,H) = -C_2\left(HD + \frac{D^2}{4}\right)$

Substituting $H = RD$

$$\underline{\text{Minimize}}\ f(D,R) = -C_2\left(RD^2 + \frac{D^2}{4}\right) = -C_2D^2\left(R + \tfrac{1}{4}\right) \qquad [10\%]$$

(b) The mass of the cylinder is proportional to its volume

$$M = \rho\frac{\pi D^2}{4}H = C_3D^2H \le M_{\max} \qquad (2.1)$$

Substituting $H = RD \quad \therefore \quad M = C_3D^2RD = C_3RD^3 \le M_{\max}$

Bounds

$$0 < R \le R_{\max}$$

$$0 < D \le D_{\max} \qquad [10\%]$$

(c)

$$\frac{\partial f}{\partial D} = -2C_2D\left(R + \tfrac{1}{4}\right)$$

$$\frac{\partial f}{\partial R} = -C_2D^2$$

These show that f reduces (improves) for non-negative values of D and R. The improvement in f will therefore be limited by the constraint on M, the upper bound on R or the upper bound on D, or more than one of these. [15%]

(d) From equation (2.1) $\quad R = \frac{M_{\max}}{C_3D^3} = \frac{C_4}{D^3}$

$$\therefore \quad f(D) = -C_2D^2\left(\frac{C_4}{D^3} + \frac{1}{4}\right) = -C_2\left(\frac{C_4}{D} + \frac{D^2}{4}\right)$$

The upper bound on R implies a lower bound on D:

$$R = \frac{C_4}{D^3} \le R_{\max} \quad \Rightarrow \quad D \ge \sqrt[3]{\frac{C_4}{R_{\max}}} = D_{\min}$$

$$D_{\min} \le D \le D_{\max}$$

Obviously, for the constraint on M to be active, $D_{\min}$ must be less than $D_{\max}$, i.e.

$$\sqrt[3]{\frac{C_4}{R_{\max}}} \le D_{\max} \qquad [20\%]$$

(e)
$$f(D) = -C_2\left(\frac{C_4}{D} + \frac{D^2}{4}\right)$$
$$\therefore \quad \frac{df}{dD} = -C_2\left(-\frac{C_4}{D^2} + \frac{D}{2}\right)$$
$$\therefore \quad \frac{d^2f}{dD^2} = -C_2\left(\frac{2C_4}{D^3} + \frac{1}{2}\right)$$

A stationary point for f exists when
$$\frac{df}{dD} = -C_2\left(-\frac{C_4}{D^2} + \frac{D}{2}\right) = 0 \quad \Rightarrow \quad D^3 = 2C_4 \quad \Rightarrow \quad D = \sqrt[3]{2C_4}$$

At this value of D
$$\frac{d^2f}{dD^2} = -C_2\left(\frac{2C_4}{2C_4} + \frac{1}{2}\right) = -\tfrac{3}{2}C_2$$

For a minimum we want $\dfrac{df}{dD} = 0$ and $\dfrac{d^2f}{dD^2} > 0$. Here $\dfrac{d^2f}{dD^2} < 0$, so no unconstrained minimum exists. [20%]

(f) The values of the gradient of f found in (c) show that for a specified value of D or R it is possible to reduce f by increasing the corresponding value of R or D. This means that the minimum will occur at either the intersection of the bounds on M and D or the intersection of the bounds on M and R. (If the constraint on M can be active.)

In the first case, $D = D_{\max}$ and $R = \dfrac{C_4}{D_{\max}{}^3}$
$$\therefore \quad f = -C_2 D^2\left(R + \tfrac{1}{4}\right) = -C_2 D_{\max}{}^2\left(\frac{C_4}{D_{\max}{}^3} + \frac{1}{4}\right)$$

In the second case, $R = R_{\max} \quad \Rightarrow \quad D = D_{\min} = \sqrt[3]{\dfrac{C_4}{R_{\max}}}$
$$\therefore \quad f = -C_2 D^2\left(R + \tfrac{1}{4}\right) = -C_2\left(\frac{C_4}{R_{\max}}\right)^{\frac{2}{3}}\left(R_{\max} + \tfrac{1}{4}\right)$$

Which is the optimum will depend on which gives the lower (more negative) value of f, which will depend on the specified values of $D_{\max}$, $R_{\max}$ and $M_{\max}$ (which affects C_4).

If $\sqrt[3]{\dfrac{C_4}{R_{\max}}} > D_{\max}$, i.e. the constraint on M cannot be active, then the optimum will be at $D = D_{\max}$, $R = R_{\max}$ with $f = -C_2 D_{\max}{}^2\left(R_{\max} + \tfrac{1}{4}\right)$. [25%]

Q3

(a) Minimise $$f(x_w,F,W)=C_f F+C_s\left[F\ln(x_f)-\ln(x_p)-W\ln(x_w)\right]$$

subject to $$h_1(x_w,F,W)=1+W-F=0$$

$$h_2(x_w,F,W)=x_p+x_w W-x_f F=0$$ [10%]

(b) The Lagrangian

$$\ell(x_w,F,W)=C_f F+C_s\left[F\ln(x_f)-\ln(x_p)-W\ln(x_w)\right]+\lambda_1(1+W-F)+\lambda_2(x_p+x_w W-x_f F)$$

For an optimum

$$\frac{\partial\ell}{\partial x_w}=-\frac{C_s W}{x_w}+\lambda_2 W=0 \quad\Rightarrow\quad \lambda_2=\frac{C_s}{x_w}$$

$$\frac{\partial\ell}{\partial W}=-C_s\ln(x_w)+\lambda_1+\lambda_2 x_w=0$$

$$\therefore\quad \lambda_1=C_s\ln(x_w)-\lambda_2 x_w=C_s\ln(x_w)-C_s=C_s\left[\ln(x_w)-1\right]$$

$$\frac{\partial\ell}{\partial F}=C_f+C_s\ln(x_f)-\lambda_1-\lambda_2 x_f=0$$

$$\therefore\quad C_f+C_s\ln(x_f)-C_s\left[\ln(x_w)-1\right]-\frac{C_s}{x_w}x_f=0$$

$$\therefore\quad \frac{C_f}{C_s}+\ln(x_f)-\ln(x_w)+1-\frac{x_f}{x_w}=0$$

$$\therefore\quad \frac{x_f}{x_w}=1+\frac{C_f}{C_s}+\ln\left(\frac{x_f}{x_w}\right)$$

The second-order conditions must also be checked.

$$\frac{\partial^2\ell}{\partial {x_w}^2}=\frac{C_s W}{{x_w}^2}$$

$$\frac{\partial^2\ell}{\partial x_w\partial W}=-\frac{C_s}{x_w}+\lambda_2=-\frac{C_s}{x_w}+\frac{C_s}{x_w}=0$$

$$\frac{\partial^2\ell}{\partial W^2}=0;\quad \frac{\partial^2\ell}{\partial F^2}=0;\quad \frac{\partial^2\ell}{\partial W\partial F}=0;\quad \frac{\partial^2\ell}{\partial F\partial x_w}=0$$

Hence $$\nabla^2\ell=\begin{bmatrix}\frac{C_s W}{{x_w}^2} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

This is clearly positive definite for positive x_w and W. Therefore we have a <u>minimum</u>. [60%]

(c) If $\dfrac{C_f}{C_s} = 1$ we need to solve $z = 2 + \ln(z)$ where $z = \dfrac{x_f}{x_w}$

This transcendental equation can be solved by an iterative scheme

$$z_{n+1} = 2 + \ln(z_n)$$

Using a first guess $z_0 = 2$, this converges to

$$z = 3.1462$$

$$\therefore x_w = \frac{x_f}{z} = \frac{0.00715}{3.1462} = 0.00227$$

$F = 1 + W$

$x_f F = x_p + x_w W \qquad \therefore \quad x_f(1+W) = x_p + x_w W$

$$\therefore \quad W(x_f - x_w) = x_p - x_f$$

$$\therefore \quad W = \frac{x_p - x_f}{x_f - x_w} = \frac{0.035 - 0.00715}{0.00715 - 0.00227} = 5.707 \text{ kg}$$

$$\therefore \quad F = 1 + W = 6.707 \text{ kg}$$

[20%]

(d) The values of the Lagrange multipliers indicate the sensitivity of the value of the optimum to relaxation of the constraints. In this case the constraint equations represent conservation laws so they cannot easily be relaxed! [10%]

4 (a) For a population $\underline{\lambda}$ moving through a Markov process governed by a transition matrix P, any distribution which is stable

i.e. $\underline{\lambda} P = \underline{\lambda}$ is a limiting distribution.

$\underline{\lambda}$ is an e-vector of eigenvalue 1 for P [10%]

(b) Every finite state space process has at least one e-vector for eigenvalue 1, & all other λ's satisfy $|\lambda| \leq 1$.

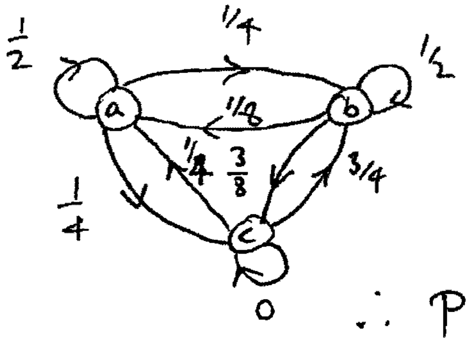
The limiting distribution will be achieved in practice if

$$\underline{\lambda_0} P^N \to \underline{\lambda} \quad \text{as } N \to \infty$$

This needs P to have no other λ's for which $|\lambda| = 1$

i.e. limiting distribution unique & no other λ's of the form $\lambda = e^{i\theta}$ which leads to periodic behaviour. [20%]

(c) State space & transitions



At any state $\sum_i P_{\text{transition}} = 1$

$\Rightarrow \quad P(a \to a) = \frac{1}{2} \qquad P(b \to b) = \frac{1}{2}$

$P(c \to c) = 0 \qquad$ Take $(1,2,3) = (a,b,c)$

$$\therefore P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & 3/8 \\ \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix}$$

[10%]

(d) Assuming an e-value of 1

$$\left.\begin{aligned} \tfrac{1}{2}a + \tfrac{b}{8} + \tfrac{c}{4} &= a \\ \tfrac{1}{4}a + \tfrac{b}{2} + \tfrac{3c}{4} &= b \\ \tfrac{1}{4}a + \tfrac{3b}{8} &= c \end{aligned}\right\} \Rightarrow \begin{aligned} -\tfrac{a}{2} + \tfrac{b}{8} + \tfrac{c}{4} &= 0 \\ \tfrac{a}{2} - b + \tfrac{3c}{2} &= 0 \\ \tfrac{a}{4} + \tfrac{3b}{8} - c &= 0 \end{aligned}$$

1st 2 $\Rightarrow -\frac{7}{8}b + \frac{7c}{4} = 0 \quad \Rightarrow b = 2c \quad = 2a$

3rd eq^n: $\frac{1}{4} + \frac{3\times 2}{8} = 1$ ✓ works

$\therefore \lambda = 1$ is e-value with e-vector $\underline{\lambda} = k\begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}1/4\\1/2\\1/4\end{bmatrix}$ limiting distr.

det P = 1/2 (-9/32) - 1/4 (-3/32) + 1/4 (3/32 - 1/8)
= -9/64 + 3/128 - 1/128 = -1/8

Now let other e-values $= \lambda_1, \lambda_2$

$1 + \lambda_1 + \lambda_2 = 1 \quad \Rightarrow \lambda_1 = \lambda_2$

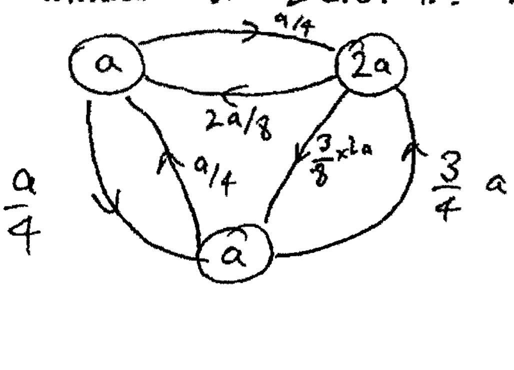
$1.\lambda_1.\lambda_2 = \det P = -\frac{1}{8} \quad \therefore \lambda_1 = \frac{1}{\sqrt{8}}, \quad \lambda_2 = -\frac{1}{\sqrt{8}}$

Gen solution for $\underline{\lambda} P^N = \underline{\alpha}\, 1^N + \underline{\beta}\left(\frac{1}{\sqrt{8}}\right)^N + \underline{\gamma}\left(-\frac{1}{\sqrt{8}}\right)^N$

$\rightarrow \underline{\alpha}$ as $N \rightarrow \infty$ since $|\lambda_1, \lambda_2| < 1$

[40%]

(e) Dynamic balance means not no. transitioning from one state to another is zero. i.e. no transitioning $b \rightarrow c$ = no transitioning $c \rightarrow b$, etc.



All flows balance
$\Rightarrow$ distribution in dynamic balance

[20%]