ENGINEERING TRIPOS PART IIA

Wednesday 1 May 2013 2 to 3.30

Module 3C6

VIBRATION

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
Data Sheet: 3C5 Dynamics and 3C6 Vibration (6 pages)

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Final version

1 (a) An oilwell drillstring can be regarded as a length L of pipe with a thin circular cross-section of external radius a and wall thickness h. The material of the pipe has shear modulus G and density ρ. Unsteady forces at the drill-bit can excite torsional vibrations in the drillstring. Regarding the top of the drillstring as being rigidly fixed while the bottom end is free, derive expressions for the mode shapes and natural frequencies of torsional vibration. Sketch the first three mode shapes. [35%]

(b) A better approximation for the top of the drillstring is that it is fixed to a rotor with polar moment of inertia J representing the drilling motor. Write down a suitable boundary condition, assuming that the rotor itself is free to rotate without constraint. Hence derive an expression whose roots give the new natural frequencies. [35%]

(c) Give a graphical construction to show where the new natural frequencies lie in relation to those found in part (a). Sketch what you think the first three mode shapes look like now. [30%]

2 (a) A xylophone bar can be regarded as a bending beam with a uniform rectangular cross-section, with free boundaries at both ends. The length of the bar is L, the width b and the thickness h, and it is made of material with Young's modulus E and density ρ. Derive an expression whose roots give the natural frequencies. [25%]

(b) Given that the first two roots of $\cos z \cosh z = 1$ are $z = 4.73$ and 7.85, find the first two natural frequencies in Hz if $L = 0.5$ m, $b = 0.05$ m, $h = 0.02$ m and the bar is made of wood with a Young's modulus $E = 20$ GPa and density $\rho = 800$ kg m^{-3}. Sketch the corresponding mode shapes. [25%]

(c) In order to obtain a better musical note from the bar, it is desired to adjust these frequencies so that they are exactly in a simple whole-number ratio. It is proposed to do this by adding a small mass M at a chosen position $x = a$, where x is the distance along the beam measured from one end. Assuming that the mode shapes do not change much as a result of the added mass, use Rayleigh's principle to show that the perturbed natural frequency Ω_n of a mode $\phi_n(x)$ with original natural frequency ω_n is given approximately by an equation of the form

$$\Omega_n^2 \approx \omega_n^2 \left\{1 + B\phi_n^2(a)\right\}$$

where B is an expression to be determined. [30%]

(d) Identify the closest whole-number ratio of frequencies to aim for. Without detailed calculations, describe a suitable strategy to achieve the desired musical tuning with minimal added mass. [20%]

3 A four-storey building is subject to earthquake loading. The system is modelled in two dimensions as shown in Fig. 1. Four floors, each of mass m, are able to move laterally only. Small displacements from equilibrium are denoted by the vector $\begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix}^T$. The floors are connected to each other by undamped walls, with lateral stiffness $k/2$. The base moves laterally with displacement x.

(a) Write the matrix equations of motion for the system. Show that displacement x of the base gives rise to an equivalent force kx applied to the first floor. [20%]

(b) Without calculations, sketch the expected mode shapes for undamped lateral vibration of the building. [20%]

(c) On a dB scale, sketch the vibration transfer function for displacement y_4 of the top floor due to the lateral displacement x of the base. [15%]

(d) Estimate the first natural frequency ω_1 with Rayleigh's Quotient using the approximate mode vector $\begin{bmatrix} 4 & 7 & 9 & 10 \end{bmatrix}^T$, which is the shape of the static deflection of the structure when equal forces are applied to each floor. [20%]

(e) If the earthquake generates motion $x = X\sin\omega t$, use the approximate mode vector from part (d) to estimate the displacement of the top floor when the frequency of excitation is half the lowest natural frequency (i.e. $\omega = \omega_1 / 2$). Comment on the sources of error in your estimate. [25%]

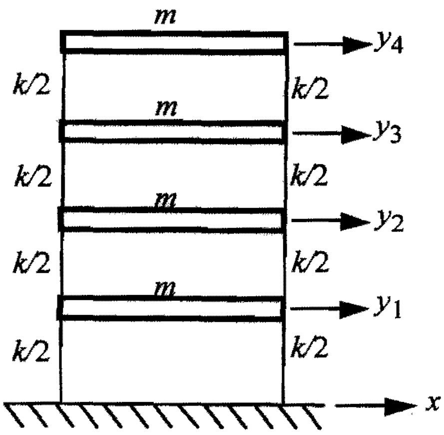

Fig. 1

4 Three masses m, $2m$ and m are connected together and to fixed supports by springs of stiffness k, as shown in Fig. 2. A spring of stiffness λk connects the outer two masses. Each mass can move in the horizontal direction only, without rotation. Small displacements from equilibrium are denoted by the coordinate vector $\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T$.

(a) Write an equation for the potential energy for small horizontal vibration of the system, and hence find the stiffness matrix. [20%]

(b) Explain how symmetry can be exploited to find one natural frequency immediately, and to recast the remaining problem in terms of two degrees of freedom. Hence find all the natural frequencies and corresponding mode shapes. [55%]

(c) Explain the influence of λ on the sequence of the natural frequencies. What value of λ makes two of the frequencies equal? If you built a physical system like this, would you in fact expect to find two equal frequencies? Justify your answer. [25%]

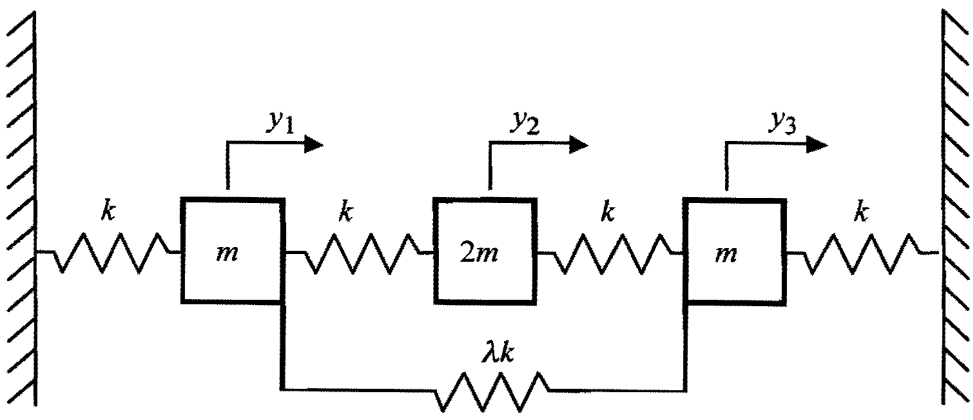

Fig. 2

END OF PAPER

Final version

Part IIA Data sheet

Module 3C5 Dynamics

Module 3C6 Vibration

DYNAMICS IN THREE DIMENSIONS

Axes fixed in direction

(a) Linear momentum for a general collection of particles m_i:

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}^{(e)}$$

where $\boldsymbol{p} = M\,\boldsymbol{v}_G$, M is the total mass, $\boldsymbol{v}_G$ is the velocity of the centre of mass and $\boldsymbol{F}^{(e)}$ the total external force applied to the system.

(b) Moment of momentum about a general point P

$$\boldsymbol{Q}^{(e)} = (\boldsymbol{r}_G - \boldsymbol{r}_P) \times \dot{\boldsymbol{p}} + \dot{\boldsymbol{h}}_G$$
$$= \dot{\boldsymbol{h}}_P + \dot{\boldsymbol{r}}_P \times \boldsymbol{p}$$

where $\boldsymbol{Q}^{(e)}$ is the total moment of external forces about P. Here, $\boldsymbol{h}_P$ and $\boldsymbol{h}_G$ are the moments of momentum about P and G respectively, so that for example

$$\boldsymbol{h}_P = \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_P) \times m_i \dot{\boldsymbol{r}}_i$$
$$= \boldsymbol{h}_G + (\boldsymbol{r}_G - \boldsymbol{r}_P) \times \boldsymbol{p}$$

where the summation is over all the mass particles making up the system.

(c) For a rigid body rotating with angular velocity $\boldsymbol{\omega}$ about a fixed point P at the origin of coordinates

$$\boldsymbol{h}_P = \int \boldsymbol{r} \times (\boldsymbol{\omega} \times \mathbf{r})\,dm = I\,\boldsymbol{\omega}$$

where the integral is taken over the volume of the body, and where

$$I = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}, \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \qquad \boldsymbol{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and

$$A = \int (y^2 + z^2)\,dm \qquad B = \int (z^2 + x^2)\,dm \qquad C = \int (x^2 + y^2)\,dm$$
$$D = \int yz\,dm \qquad E = \int zx\,dm \qquad F = \int xy\,dm$$

where all integrals are taken over the volume of the body.

Axes rotating with angular velocity $\boldsymbol{\Omega}$

Time derivatives of vectors must be replaced by the "rotating frame" form, so that for example

$$\dot{\boldsymbol{p}} + \boldsymbol{\Omega} \times \boldsymbol{p} = \boldsymbol{F}^{(e)}$$

where the time derivative is evaluated in the moving reference frame.

When the rate of change of the position vector $\boldsymbol{r}$ is needed, as in 1(b) above, it is usually easiest to calculate velocity components directly in the required directions of the axes. Application of the general formula needs an extra term unless the origin of the frame is fixed.

Euler's dynamic equations (governing the angular motion of a rigid body)

(a) Body-fixed reference frame:

$$A\,\dot{\omega}_1 - (B - C)\,\omega_2\,\omega_3 = Q_1$$
$$B\,\dot{\omega}_2 - (C - A)\,\omega_3\,\omega_1 = Q_2$$
$$C\,\dot{\omega}_3 - (A - B)\,\omega_1\,\omega_2 = Q_3$$

where A, B and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes aligned with the principal axes of inertia of the body at P.

(b) Non-body-fixed reference frame for axisymmetric bodies (the "Gyroscope equations"):

$$A\,\dot{\Omega}_1 - (A\,\Omega_3 - C\,\omega_3)\,\Omega_2 = Q_1$$
$$A\,\dot{\Omega}_2 + (A\,\Omega_3 - C\,\omega_3)\,\Omega_1 = Q_2$$
$$C\,\dot{\omega}_3 = Q_3$$

where A, A and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes such that ω_3 and Q_3 are aligned with the symmetry axis of the body. The reference frame (not fixed in the body) rotates with angular velocity $\boldsymbol{\Omega} = [\Omega_1, \Omega_2, \Omega_3]$ with $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$.

Lagrange's equations

For a holonomic system with generalised coordinates q_i

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where T is the total kinetic energy, V is the total potential energy, and Q_i are the non-conservative generalised forces.

VIBRATION MODES AND RESPONSE

Discrete systems

1. The forced vibration of an N-degree-of-freedom system with mass matrix M and stiffness matrix K (both symmetric and positive definite) is

$$M\,\ddot{\underline{y}} + K\,\underline{y} = \underline{f}$$

where $\underline{y}$ is the vector of generalised displacements and $\underline{f}$ is the vector of generalised forces.

2. Kinetic energy

$$T = \frac{1}{2}\dot{\underline{y}}^t M \dot{\underline{y}}$$

Potential energy

$$V = \frac{1}{2}\underline{y}^t K \underline{y}$$

3. The natural frequencies ω_n and corresponding mode shape vectors $\underline{u}^{(n)}$ satisfy

$$K\underline{u}^{(n)} = \omega_n{}^2 M\underline{u}^{(n)} .$$

4. Orthogonality and normalisation

$$\underline{u}^{(j)t} M \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

$$\underline{u}^{(j)t} K \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ \omega_n^2, & j = k \end{cases}$$

5. General response

The general response of the system can be written as a sum of modal responses

$$\underline{y}(t) = \sum_{j=1}^{N} q_j(t)\,\underline{u}^{(j)} = U\underline{q}(t)$$

where U is a matrix whose N columns are the normalised eigenvectors $\underline{u}^{(j)}$ and q_j can be thought of as the "quantity" of the jth mode.

Continuous systems

The forced vibration of a continuous system is determined by solving a partial differential equation: see p. 6 for examples.

$$T = \frac{1}{2}\int \dot{u}^2 dm$$

where the integral is with respect to mass (similar to moments and products of inertia).

See p. 4 for examples.

The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see p. 4) and boundary conditions, assuming harmonic time dependence.

$$\int u_j(x)\,u_k(x)\,dm = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

The general response of the system can be written as a sum of modal responses

$$w(x,t) = \sum_j q_j(t)\,u_j(x)$$

where $w(x,t)$ is the displacement and q_j can be thought of as the "quantity" of the jth mode.

6. Modal coordinates $\underline{q}$ satisfy

$$\ddot{\underline{q}}+\left[\operatorname{diag}(\omega_j^2)\right]\underline{q}=\underline{Q}$$

where $\underline{y}=U\underline{q}$ and the modal force vector

$$\underline{Q}=U^t\underline{f}\ .$$

Each modal amplitude $q_j(t)$ satisfies

$$\ddot{q}_j+\omega_j^2\,q_j=Q_j$$

where $Q_j=\int f(x,t)\,u_j(x)\,dm$ and $f(x,t)$ is the external applied force distribution.

7. Frequency response function

For input generalised force f_j at frequency ω and measured generalised displacement y_k the transfer function is

$$H(j,k,\omega)=\frac{y_k}{f_j}=\sum_{n=1}^{N}\frac{u_j^{(n)}u_k^{(n)}}{\omega_n{}^2-\omega^2}$$

(with no damping), or

$$H(j,k,\omega)=\frac{y_k}{f_j}\approx\sum_{n=1}^{N}\frac{u_j^{(n)}u_k^{(n)}}{\omega_n{}^2+2i\omega\omega_n\zeta_n-\omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

For force F at frequency ω applied at point x, and displacement w measured at point y, the transfer function is

$$H(x,y,\omega)=\frac{w}{F}=\sum_n\frac{u_n(x)\,u_n(y)}{\omega_n{}^2-\omega^2}$$

(with no damping), or

$$H(x,y,\omega)=\frac{w}{F}\approx\sum_n\frac{u_n(x)\,u_n(y)}{\omega_n{}^2+2i\omega\omega_n\zeta_n-\omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

8. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)}u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with low modal overlap, if the factor $u_n(x)\,u_n(y)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

9. Impulse response

For a unit impulsive generalised force $f_j=\delta(t)$ the measured response y_k is given by

$$g(j,k,t)=y_k(t)=\sum_{n=1}^{N}\frac{u_j^{(n)}u_k^{(n)}}{\omega_n}\sin\omega_n t$$

for $t\geq 0$ (with no damping), or

$$g(j,k,t)\approx\sum_{n=1}^{N}\frac{u_j^{(n)}u_k^{(n)}}{\omega_n}\sin\omega_n t\ e^{-\omega_n\zeta_n t}$$

for $t\geq 0$ (with small damping).

For a unit impulse applied at $t=0$ at point x, the response at point y is

$$g(x,y,t)=\sum_n\frac{u_n(x)\,u_n(y)}{\omega_n}\sin\omega_n t$$

for $t\geq 0$ (with no damping), or

$$g(x,y,t)\approx\sum_n\frac{u_n(x)\,u_n(y)}{\omega_n}\sin\omega_n t\ e^{-\omega_n\zeta_n t}$$

for $t\geq 0$ (with small damping).

10. Step response

For a unit step generalised force $f_j = \begin{cases} 0 & t < 0 \\ 1 & t \ge 0 \end{cases}$ the measured response y_k is given by

$$h(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - \cos\omega_n t\right]$$

for $t \ge 0$ (with no damping), or

$$h(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - \cos\omega_n t \, e^{-\omega_n \zeta_n t}\right]$$

for $t \ge 0$ (with small damping).

For a unit step force applied at $t = 0$ at point x, the response at point y is

$$h(x,y,t) = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2} \left[1 - \cos\omega_n t\right]$$

for $t \ge 0$ (with no damping), or

$$h(t) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2} \left[1 - \cos\omega_n t \, e^{-\omega_n \zeta_n t}\right]$$

for $t \ge 0$ (with small damping).

Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is $\frac{V}{\tilde{T}} = \frac{\underline{y}^t K \underline{y}}{\underline{y}^t M \underline{y}}$ where $\underline{y}$ is the vector of generalised coordinates, M is the mass matrix and K is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions on p. 6.

If this quantity is evaluated with any vector $\underline{y}$, the result will be

(1) $\ge$ the smallest squared frequency;

(2) $\le$ the largest squared frequency;

(3) a good approximation to ω_k^2 if $\underline{y}$ is an approximation to $\underline{u}^{(k)}$.

(Formally, $\frac{V}{\tilde{T}}$ is *stationary* near each mode.)

GOVERNING EQUATIONS FOR CONTINUOUS SYSTEMS

Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $w(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\dfrac{\partial^2 w}{\partial t^2} - P\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}P\displaystyle\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}m\displaystyle\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $f(x,t)$ per unit length.
Polar moment of area is $J = (\pi/2)(a^4 - b^4)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\dfrac{\partial^2 \theta}{\partial t^2} - GJ\dfrac{\partial^2 \theta}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}GJ\displaystyle\int\left(\dfrac{\partial \theta}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho J\displaystyle\int\left(\dfrac{\partial \theta}{\partial t}\right)^2 dx$

Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $w(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 w}{\partial t^2} - EA\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}EA\displaystyle\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\displaystyle\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $w(x,t)$, applied transverse force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 w}{\partial t^2} + EI\dfrac{\partial^4 w}{\partial x^4} = f(x,t)$	$V = \dfrac{1}{2}EI\displaystyle\int\left(\dfrac{\partial^2 w}{\partial x^2}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\displaystyle\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.