ENGINEERING TRIPOS PART IIA

Wednesday 24 April 2013 9.30 to 11

Module 3C7

MECHANICS OF SOLIDS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: 3C7 formula sheet (2 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Books
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Version: FINAL

1 It has been suggested that stresses derived from the Airy Stress Function

$$\phi(r,\theta) = -Cr\theta\cos\theta$$

may be suitable to describe the stresses within the tapered cantilever shown in Fig. 1, in a suitable polar coordinate system (r,θ). There is a small angle of 2α between the top and bottom faces of the cantilever. A transverse downward load P is applied at its tip.

(a) Define the appropriate polar coordinate system and hence calculate the stresses within the cantilever defined by ϕ, and show that they satisfy the boundary conditions on the free edges of the cantilever. [20%]

(b) Calculate the value of C to satisfy equilibrium with the applied load. [30%]

(c) A Cartesian coordinate system is defined with its origin at the tip of the cantilever, with the x-axis horizontal along the axis of the cantilever, and y-axis vertically up. Show that the shear stress in the cantilever are given by

$$\sigma_{xy} = \frac{2Cxy^2}{\left(x^2+y^2\right)^2}$$

Comment on how the distribution of shear stress differs from that expected in a prismatic beam. [50%]

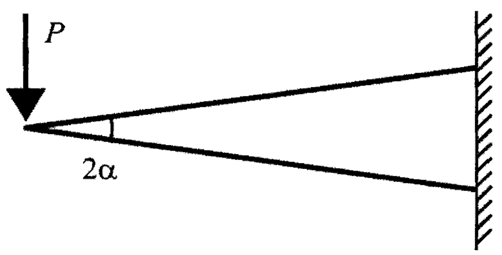

Fig. 1

2 Part of the containment structure of a nuclear reactor consists of a steel disc of diameter $D = 4\,\text{m}$ and thickness $0.2\,\text{m}$. The steel has a Young modulus of $E = 210\,\text{GPa}$, a Poisson ratio of 0.3, a coefficient of thermal expansion of $\alpha = 12 \times 10^{-6}\,\text{K}^{-1}$ and a uniaxial yield strength of 350 MPa.

The disc is stress-free at a uniform temperature of $T_0 = 300\,\text{K}$. During operation, the temperature distribution is axisymmetric, and is given as a function of radial distance by

$$T(r) = 2T_0\left[1 - \left(\frac{r}{D}\right)^2\right]$$

where r is the radial distance.

(a) Assuming that the disc is radially unconstrained:

(i) Calculate the stress distribution in the disk, showing that

$$\sigma_{\theta\theta} = -\frac{E\alpha T_0}{8} + \frac{3E\alpha T_0}{2}\left(\frac{r}{D}\right)^2$$

[30%]

(ii) Calculate the safety factor against yield, assuming a Tresca yield criterion. [20%]

(b) In an alternative scenario, the surrounding structure completely constrains the edge of the disk. Calculate the change in stress distribution from the previous case, and the safety factor against yield in this new scenario. [50%]

3 Figure 2 shows a proposed thickness reducing operation. A wide sheet of unit thickness is drawn at speed v between two smooth flat dies by the application of the tensile stress σ to the emerging material which is of thickness s. The line AB indicates a tangential velocity discontinuity.

(a) Show that the relation between the upper die angle α and the distance x, as defined in the figure, is given by:

$$\sin\alpha = \frac{\sqrt{1+x^2-s^2}-xs}{1+x^2}$$

[25%]

(b) The process is well lubricated so that there is negligible friction between the deforming material and the dies. Use an Upper Bound analysis to find an expression for the drawing stress σ in terms of the dimensions x and s and the material shear yield stress k. [25%]

(c) For a given value of s determine the value of the dimension x which will minimise the drawing stress. Demonstrate that this condition corresponds to a value of α such that

$$\sin\alpha = \frac{1-s^2}{1+s^2}$$

[25%]

(d) Estimate the maximum possible reduction in the area, i.e. the minimum value of s, that can be accomplished by this process for a Tresca material. Find the corresponding value of α. [25%]

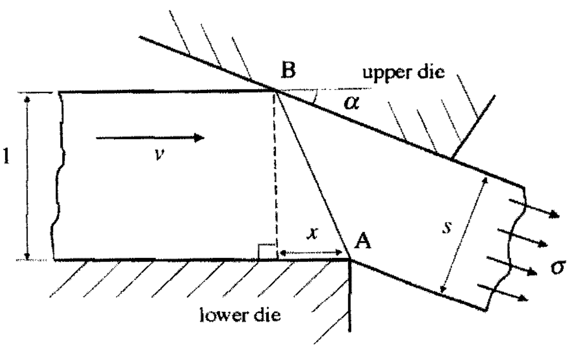

Fig. 2

4 It is suggested that for tensile specimens that are cut in different directions from a thin sheet of material and tested in simple tension that the Stassi yield criterion may be more appropriate than the von Mises criterion. The Stassi yield criterion can be written as

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 + 2\sigma_t(\rho - 1)(\sigma_1 + \sigma_2 + \sigma_3) = 2\rho\sigma_t^2$$

where σ_1, σ_2, σ_3 are the principal stresses and σ_t and ρ are material constants.

It is found that the material response changes from elastic to plastic at a characteristic stress Y irrespective of the orientation of the specimens and it is suggested that $\rho = 3$.

(a) Show that for this case it follows that $\sigma_t = Y$. [25%]

(b) To test the Stassi hypothesis it is proposed to carry out further tests examining the response of the material in simple compression and in biaxial tension with $\sigma_1 = \sigma_2$. For each of these tests investigate the differences which might be expected in the observations of the stresses at yield if the hypothesis is correct rather than the material behaving according to von Mises. [25%]

(c) A thin tube is made of the material and tested in pure torsion and the shear stress at yield is found to be k. How is the value of k related to that of Y assuming the Stassi criterion applies? [25%]

(d) Sketch the yield curve for bi-axial conditions according to the Stassi criterion in the first and fourth quadrants of a Cartesian plot of σ_1/Y and σ_2/Y. On the yield locus locate the points representing the conditions of the tests specified in parts (b) and (c). [25%]

END OF PAPER

Module 3C7: Mechanics of Solids

ELASTICITY and PLASTICITY FORMULAE

1. Axi-symmetric deformation : discs, tubes and spheres

	Discs and tubes	Spheres
Equilibrium	$\sigma_{\theta\theta} = \dfrac{d(r\sigma_{rr})}{dr} + \rho\omega^2 r^2$	$\sigma_{\theta\theta} = \dfrac{1}{2r}\dfrac{d(r^2\sigma_{rr})}{dr}$
Lamé's equations (in elasticity)	$\sigma_{rr} = A - \dfrac{B}{r^2} - \dfrac{3+\nu}{8}\rho\omega^2 r^2 - \dfrac{E\alpha}{r^2}\int_c^r rT\,dr$	$\sigma_{rr} = A - \dfrac{B}{r^3}$
	$\sigma_{\theta\theta} = A + \dfrac{B}{r^2} - \dfrac{1+3\nu}{8}\rho\omega^2 r^2 + \dfrac{E\alpha}{r^2}\int_c^r rT\,dr - E\alpha T$	$\sigma_{\theta\theta} = A + \dfrac{B}{2r^3}$

2. Plane stress and plane strain

Plane strain elastic constants $\bar{E} = \dfrac{E}{1-\nu^2}$; $\bar{\nu} = \dfrac{\nu}{1-\nu}$; $\bar{\alpha} = \alpha(1+\nu)$

	Cartesian coordinates	Polar coordinates
Strains	$\varepsilon_{xx} = \dfrac{\partial u}{\partial x}$	$\varepsilon_{rr} = \dfrac{\partial u}{\partial r}$
	$\varepsilon_{yy} = \dfrac{\partial v}{\partial y}$	$\varepsilon_{\theta\theta} = \dfrac{u}{r} + \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$
	$\gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$	$\gamma_{r\theta} = \dfrac{\partial v}{\partial r} + \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r}$
Compatibility	$\dfrac{\partial^2\gamma_{xy}}{\partial x\partial y} = \dfrac{\partial^2\varepsilon_{xx}}{\partial y^2} + \dfrac{\partial^2\varepsilon_{yy}}{\partial x^2}$	$\dfrac{\partial}{\partial r}\left\{r\dfrac{\partial\gamma_{r\theta}}{\partial\theta}\right\} = \dfrac{\partial}{\partial r}\left\{r^2\dfrac{\partial\varepsilon_{\theta\theta}}{\partial r}\right\} - r\dfrac{\partial\varepsilon_{rr}}{\partial r} + \dfrac{\partial^2\varepsilon_{rr}}{\partial\theta^2}$
or (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}(\sigma_{xx} + \sigma_{yy}) = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right\}(\sigma_{rr} + \sigma_{\theta\theta}) = 0$
Equilibrium	$\dfrac{\partial\sigma_{xx}}{\partial x} + \dfrac{\partial\sigma_{xy}}{\partial y} = 0$	$\dfrac{\partial}{\partial r}(r\sigma_{rr}) + \dfrac{\partial\sigma_{r\theta}}{\partial\theta} - \sigma_{\theta\theta} = 0$
	$\dfrac{\partial\sigma_{yy}}{\partial y} + \dfrac{\partial\sigma_{xy}}{\partial x} = 0$	$\dfrac{\partial\sigma_{\theta\theta}}{\partial\theta} + \dfrac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0$
$\nabla^4\phi = 0$ (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}\left\{\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2}\right\} = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right\} \times \left\{\dfrac{\partial^2\phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}\right\} = 0$
Airy Stress Function	$\sigma_{xx} = \dfrac{\partial^2\phi}{\partial y^2}$	$\sigma_{rr} = \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}$
	$\sigma_{yy} = \dfrac{\partial^2\phi}{\partial x^2}$	$\sigma_{\theta\theta} = \dfrac{\partial^2\phi}{\partial r^2}$
	$\sigma_{xy} = -\dfrac{\partial^2\phi}{\partial x\partial y}$	$\sigma_{r\theta} = -\dfrac{\partial}{\partial r}\left\{\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}\right\}$

3. Torsion of prismatic bars

Prandtl stress function: $\sigma_{zx}\ (= \tau_x) = \dfrac{\partial \psi}{\partial y}$, $\sigma_{zy}\ (= \tau_y) = -\dfrac{\partial \psi}{\partial x}$

Equilibrium: $T = 2\int_A \psi \, dA$

Governing equation for elastic torsion: $\nabla^2 \psi = -2G\beta$ where β is the angle of twist per unit length.

4. Total potential energy of a body

$$\Pi = U - W$$

where $U = \frac{1}{2}\int_V \underset{\sim}{\varepsilon}^T [D]\, \underset{\sim}{\varepsilon}\, dV$, $W = \underset{\sim}{P}^{\,T} \underset{\sim}{u}$ and $[D]$ is the elastic stiffness matrix.

5. Principal stresses and stress invariants

Values of the principal stresses, σ_P, can be obtained from the equation

$$\begin{vmatrix} \sigma_{xx} - \sigma_P & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \sigma_P & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \sigma_P \end{vmatrix} = 0$$

This is equivalent to a cubic equation whose roots are the values of the 3 principal stresses, i.e. the possible values of σ_P.

Expanding: $\sigma_P^3 - I_1 \sigma_P^2 + I_2 \sigma_P - I_3 = 0$ where $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$,

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} \quad \text{and} \quad I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}.$$

6. Equivalent stress and strain

Equivalent stress $\bar{\sigma} = \sqrt{\frac{1}{2}}\ \{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\}^{1/2}$

Equivalent strain increment $d\bar{\varepsilon} = \sqrt{\frac{2}{3}}\ \{d\varepsilon_1^2 + d\varepsilon_2^2 + d\varepsilon_3^2\}^{1/2}$

7. Yield criteria and flow rules

Tresca

Material yields when maximum value of $|\sigma_1 - \sigma_2|$, $|\sigma_2 - \sigma_3|$ or $|\sigma_3 - \sigma_1| = Y = 2k$, and then,

if σ_3 is the intermediate stress, $d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = \lambda(1 : -1 : 0)$ where $\lambda \neq 0$.

von Mises

Material yields when, $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2 = 6k^2$, and then

$$\frac{d\varepsilon_1}{\sigma'_1} = \frac{d\varepsilon_2}{\sigma'_2} = \frac{d\varepsilon_3}{\sigma'_3} = \frac{d\varepsilon_1 - d\varepsilon_2}{\sigma_1 - \sigma_2} = \frac{d\varepsilon_2 - d\varepsilon_3}{\sigma_2 - \sigma_3} = \frac{d\varepsilon_3 - d\varepsilon_1}{\sigma_3 - \sigma_1} = \lambda = \frac{3}{2}\frac{d\bar{\varepsilon}}{\bar{\sigma}}.$$