ENGINEERING TRIPOS PART IIA

Thursday 2 May 2013 2.30 to 4

Module 3D7

FINITE ELEMENT METHODS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: 3D7 Data Sheet (3 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Books
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Version: FINAL

1 Wave propagation in a one-dimensional elastic bar is governed by

$$\rho \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{dx}\left(E\frac{\partial u}{\partial x}\right) = f$$

where the density ρ is constant, u is the displacement, $E = E_0 + \beta x$ is Young's modulus, where E_0 and β are positive constants, and f is prescribed.

(a) For a linear element running from $x = 0$ to $x = l$, compute the element mass and stiffness matrices. [40%]

(b) The semi-discrete formulation of this problem reads

$$\boldsymbol{M}\ddot{\boldsymbol{a}} + \boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}$$

By generalising the scheme $a_{n+1} - a_n = \Delta t \dot{a}_{n+1}$, formulate a fully discrete problem. [30%]

(c) The wave propagation problem can also be formulated as two coupled first-order-in-time problems. Write down the two coupled equations, in which u is one of the unknowns, and devise a fully discrete formulation using the scheme $a_{n+1} - a_n = \Delta t \dot{a}_{n+1}$. [30%]

2 Two finite element simulations of steady heat conduction in a cube are performed using a structured mesh of trilinear hexahedral elements. Each element has eight nodes and forms a cube, and all elements in a given mesh have the same size. Case A uses n elements and Case B uses $2n$ elements, where n is large.

(a) How many non-zero entries would you expect in most rows of the global stiffness matrix for Case A and Case B? [20%]

(b) Approximate the increase in time and memory required to build the global stiffness matrix for Case B compared to Case A. [20%]

(c) Cases A and B are solved using an LU solver, and then again with a multigrid preconditioned solver. Based on the ideal theoretical complexities of the LU and multigrid solvers, for each solver estimate the increase in solver time in going from Case A to Case B. [20%]

(d) If the error in the L^2 norm of the solution is proportional to Ch^2, where h is the length of an element edge and C is a problem constant that does not depend on h, estimate the reduction in the error for Case B compared to Case A. [20%]

(e) Following the steady simulations, an unsteady heat conduction problem is solved for Cases A and B using a conditionally stable time stepping scheme that does not require a system of linear equations to be solved. If both cases are run at their critical time step, estimate the relative difference in the time step between the two cases. If both cases are run to the same terminal time, is the relative difference in the time step the same as the relative difference in the total time cost for the two cases? [20%]

3 (a) Figure 1(a) shows two finite elements to be used as transition elements. Give analytical expressions for the shape functions corresponding to the following nodes and elements:

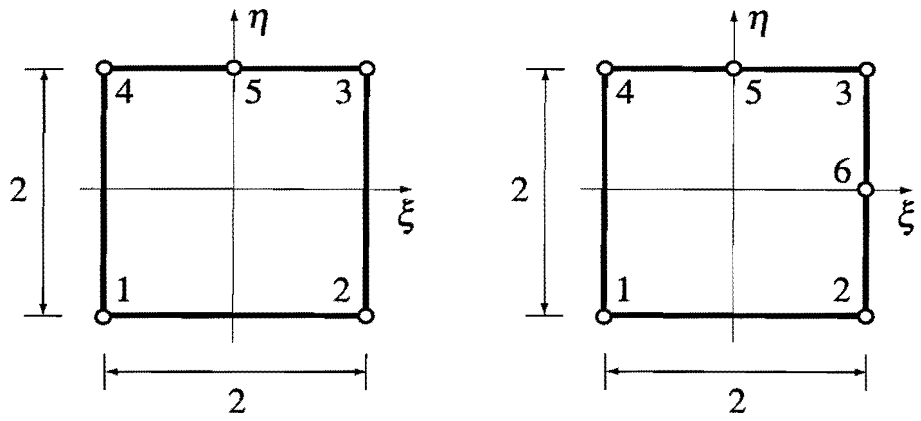
(i) node 3 of the five-node quadrilateral element [20%]

(ii) nodes 3 and 4 of the six-node quadrilateral element [20%]

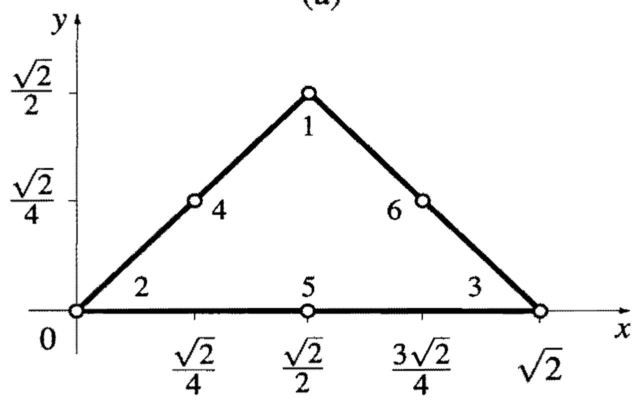
(b) Figure 1(b) shows a six-noded finite element. For a problem with a prescribed normal heat flux and a domain heat source, the normal heat flux on the edge between nodes 1 and 2 is $\bar{q} = 1$ and is zero on the other two edges, and the domain heat source is $s = 2$.

(i) Compute the element flux vector resulting from the normal heat flux $\bar{q}$. [30%]

(ii) Compute the second component of the element source vector resulting from the domain heat source term s. [30%]



(a)



(b)

Fig. 1

4 (a) Figure 2 shows a four-noded parent element and a particular isoparametric element. The point P in the isoparametric element corresponds to a quadrature point at $(1/\sqrt{3}, 1/\sqrt{3})$ in the parent domain.

(i) Compute the Jacobian matrix

$$\vec{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

at point P. [50 %]

(ii) Compute the physical derivatives of the shape function corresponding to node 3 at point P, i.e.

$$\frac{\partial N_3}{\partial x} \quad \text{and} \quad \frac{\partial N_3}{\partial y}$$

[20 %]

(b) On a domain Ω with boundary Γ consider the two-dimensional heat equation

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + s = 0$$

where T is the temperature, s is a domain heat source and k is the constant heat conductivity. The boundary Γ is subjected to the combined conduction and convection condition

$$k\frac{\partial T}{\partial x}n_x + k\frac{\partial T}{\partial y}n_y + \beta(T - T_\infty) = \bar{q}$$

where n_x and n_y are the components of the outward unit normal vector to the boundary, β is a constant heat transfer coefficient, T_∞ is the constant ambient temperature and $\bar{q}$ is the prescribed normal heat flux. Derive the weak form of this boundary value problem. [30%]

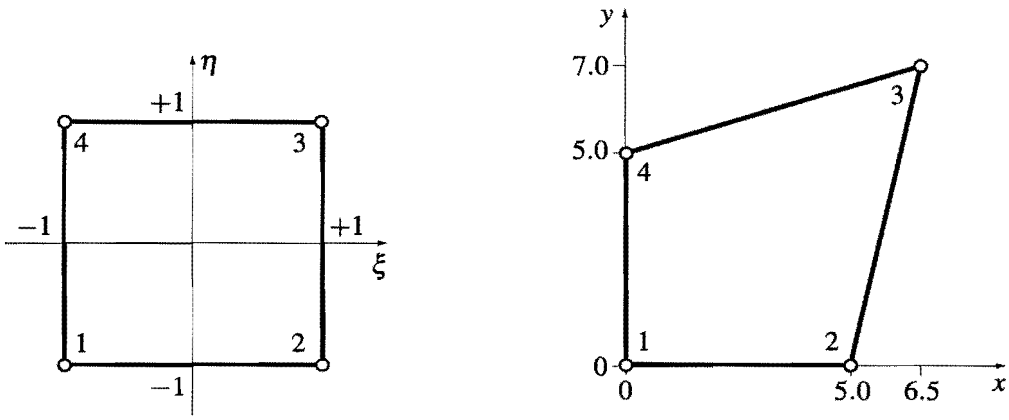


Fig. 2

END OF PAPER

Version: FINAL