ENGINEERING TRIPOS PART IIA

Tuesday 30 April 2013 9.30 to 11

Module 3F1

SIGNALS AND SYSTEMS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

~~version~~ 3
Final corrected
on master

1 A discrete-time system with input sequence $\{u_k\}$ and output sequence $\{y_k\}$, has transfer function $G(z) = \dfrac{(z+1)}{z(z-1)}$.

(a) Calculate the step response of this system, and check your answer is consistent with the initial value theorem and/or the final value theorem if either one applies. [15%]

(b) Write down a difference equation with this transfer function and check the first three values of the step response calculated above agree with the corresponding solution of the difference equation. [15%]

(c) Show that

$$G(e^{j\theta}) = \frac{-j}{\tan(\theta/2)}\, e^{-j\theta}.$$

[15%]

(d) Sketch the Nyquist diagram for $G(z)$. This system is now connected in a standard unity gain negative feedback arrangement with a precompensator with a constant gain, K. Using the Nyquist stability criterion determine what values of K give closed-loop stability. [20%]

(e) For $K = 1/2$ and the external reference signal, $r_k = \cos(\omega kT)$, determine the behaviour of the error, $e_k = r_k - y_k$ as k becomes large for the three cases, $\omega T = 0,\ \pi/4,\ \pi$.

[20%]

(f) If the input to $G(z)$ is $u_k = (-1)^k$ for $k \geq 0$, calculate the output and comment on its relation to the frequency response. [15%]

2 (a) A linear discrete-time system with input sequence $\{u_k\}$ and output sequence $\{y_k\}$, has pulse response sequence $\{g_k\}$ and transfer function $G(z)$.

(i) Show that if

$$\sum_{k=0}^{\infty} |g_k| = M < \infty \qquad (1)$$

then bounded inputs will produce bounded outputs. [20%]

(ii) In the case $G(z) = \dfrac{1}{(z^2+1)}$ show that (1) does not hold and that there exists a bounded input that gives an unbounded output sequence. [30%]

(b) An ergodic random signal, $X(t)$, is passed through a linear system with impulse response, $h(t)$, and generates an output signal, $Y(t)$. The cross correlation function between $X(t)$ and $Y(t)$ is defined to be

$$r_{XY}(\tau) = E[X(t)\,Y(t+\tau)]$$

where $E[\,]$ is the expectation operator over time, t.

(i) Derive an expression, showing that $r_{XY}(\tau)$ is related to the input autocorrelation function $r_{XX}(\tau)$ by convolution. [20%]

(ii) State how $S_X(\omega)$, the power spectrum of X, and $S_{XY}(\omega)$, the cross-spectrum of X with Y, relate to $r_{XX}(\tau)$ and $r_{XY}(\tau)$. [15%]

(iii) Hence obtain an expression for the system frequency response, $H(\omega)$, assuming that $h(t)$ is unmeasurable in normal operation of the system, but that it is possible to obtain $r_{XX}(\tau)$ and $r_{XY}(\tau)$ by analysing X and Y. [15%]

3 (a) The characteristic function of a random variable, X, is defined by the expression

$$\Phi_X(u) = E[e^{juX}]$$

Show that Φ_X is related to the Fourier Transform of the probability density function, $f_X(x)$, of the random variable, if x replaces the usual variable, time. [20%]

(b) If Y is the sum of two random variables, X_1 and X_2, state how Φ_Y relates to Φ_{X_1} and Φ_{X_2}, the characteristic functions of X_1 and X_2, giving reasons. [20%]

(c) If the probability density functions of X_1 and X_2 are given by

$$f_1(x) = \begin{cases} \frac{1}{A}\left(1 - \frac{|x|}{A}\right) & \text{if } |x| \leq A \\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} \frac{1}{A} & \text{if } 0 \leq x \leq A \\ 0 & \text{otherwise}, \end{cases}$$

determine $\Phi_Y(u)$ when $Y = X_1 + X_2$. [30%]

(d) Explain in principle how the n^{th}-order moment of $f_Y(y)$ can be calculated from a derivative of $\Phi_Y(u)$, and, as an example, calculate the first-order moment of Y, as defined in part (c). [30%]

4 The joint probability distribution for a first-order Markov message source X, emitting symbols A, B or C at times $(n-1)$ and n is given by the following table:

$P(X_{n-1}, X_n)$:

X_n \ X_{n-1}	A	B	C
A	0.2	0.05	0.05
B	0.05	0.3	0.05
C	0.05	0.05	0.2

(a) Show that this is a valid joint probability distribution, and determine the probabilities of states A, B and C in a long stream of symbols from this source, assuming the above table applies to symbols emitted at all integer times, n. [15%]

(b) From the above table derive the table for a *conditional* probability distribution for symbol X_n, given symbol X_{n-1}, and calculate the mutual information between consecutive symbols in the long stream. [25%]

(c) It is desired to encode blocks of 4 consecutive symbols from this source in a binary code, where each block is encoded independently of other blocks. Estimate the average number of bits needed to code each block of 4 symbols, assuming an ideal variable length encoder could be designed. [15%]

(d) Design a Huffman coder of order 2 for this source and hence estimate the mean number of bits it would use to encode each block of 4 symbols from part (c). Hence calculate the efficiency of this code. [25%]

(e) Briefly discuss any improved encoding techniques that might be employed for this data source. [20%]

END OF PAPER