4A12 Turbulence Solutions 2003

Question 1

a) $\nabla^2 u_y = \frac{1}{r}\frac{d}{dr}\left[r\frac{du_y}{dr}\right] = \frac{\partial p}{\partial y} = a$

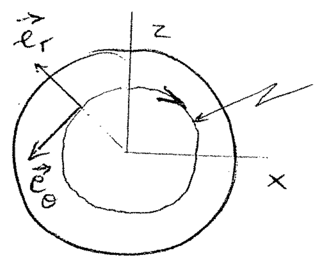
$$u_y = \frac{a}{4}\left(r^2 - \frac{D^2}{4}\right)$$

$$\bar{v} = \frac{1}{\pi D^2/4}\int_0^{D/2} 2\pi\, r\, u_y\, dr = -\frac{a D^2}{32}$$

hence $u_y = -\frac{8\bar{v}}{D^2}\left(r^2 - \frac{D^2}{4}\right)$ [Given in the question]

Remark: $Re = \frac{\bar{v} D}{\nu} \simeq \frac{10^{-1}\, 10^{-2}}{10^{-6}} = 10^3$ laminar

a b) Vorticity $\omega_r = \omega_y = 0$ $\quad \omega_\theta = \frac{du_y}{dr} = -16\,\bar{v}\frac{r}{D^2}$



[30%]

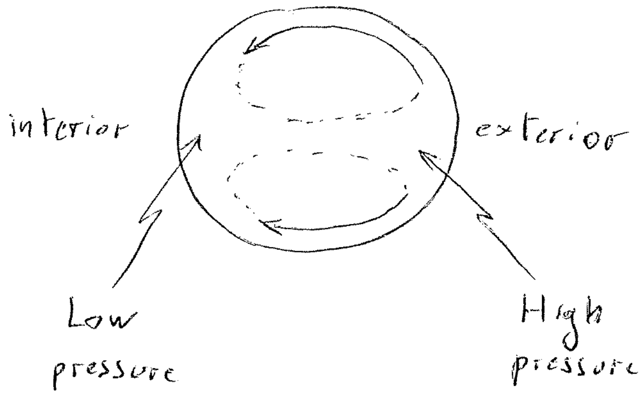
b) (i) When $z > 0$ $\quad \omega_x > 0$ in section A-A'
When $z < 0$ $\quad \omega_x < 0$ in section A-A'

Inviscid vortex equation on a short length $\frac{D\omega_x}{Dt} = (\vec{\omega}\cdot\vec{\nabla})u_x$

Both for $z>0$ and $z<0$ $(\vec{\omega}\cdot\vec{\nabla})u_x$ is strengthening ω_x. So in section B-B' we still have
$\omega_x > 0$ when $z > 0$
$\omega_x < 0$ when $z < 0$

This explains the recirculations observed. [25%]

(ii) In the bend, centrifugal forces are balanced by pressure



This pressure difference causes a flow near the boundary where centrifugal forces $\rho \frac{v^2}{R}$ are weak due to the no-slip condition. [25%]

c) Length of recirculation

Vorticity equation in the x-direction

$$\frac{D\omega_x}{Dt} = (\vec{\omega}.\vec{\nabla})\, v_x + \nu \nabla^2 \omega_x$$

Vortex stretching is very weak after the bend. Vorticity in the x direction diffuses downstream until it is homogeneous $\omega_x = 0$

$$v_x \frac{\partial \omega_x}{\partial x} \simeq \nu \nabla^2 \omega_x$$

convection downstream $\quad \bar{v}\,\frac{\omega_x}{L} \quad \sim \quad \nu \frac{\omega_x}{(D/2)^2} \quad$ diffusion within a cross-section

Diffusion has killed the recirculation when [20%]

$L \sim \frac{D^2 \bar{v}}{4\nu}$ about 2m $\quad \frac{L}{D} \sim \frac{Re}{4}$

Question 2

a) $(-\alpha x, 0, \alpha z)$ is curl free and divergence free hence v must carry the vorticity and be also divergence free. [20%]

b) Assuming the same $e^{i(k_o x + k_o y)}$ form

$v_x = B \, e^{i(k_o x + k_o y)}$

$v_y = C \, e^{i(k_o x + k_o y)}$ $\qquad \vec{\nabla}.\vec{v} = [ik_o B + ik_o C] \, e^{i(k_o x + k_o y)}$

$\longrightarrow C = -B$

2D flow, hence ω_z only $\qquad \omega_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}$

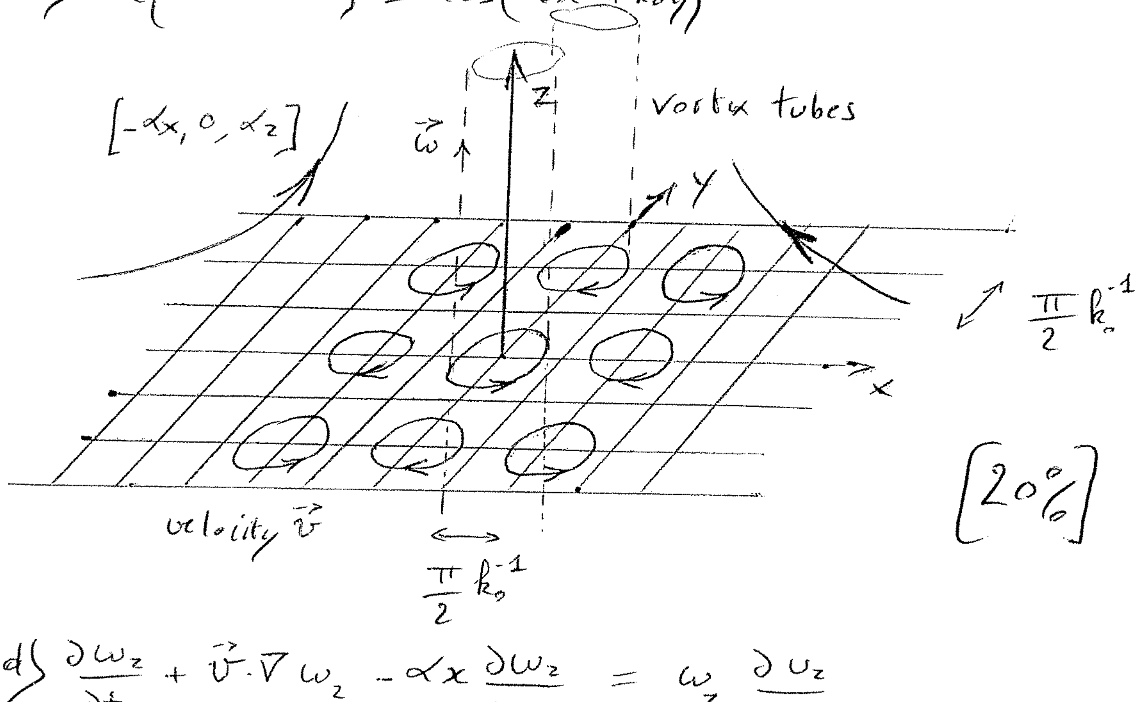
$\omega_z = (ik_o C - ik_o B) \, e^{i(k_o x + k_o y)}$

$= -2k_o B \, e^{i(k_o x + k_o y + \frac{\pi}{2})}$

Hence $v_x = -\frac{A_o}{2k_o} \mathrm{Re}\left[e^{i(k_o x + k_o y)}\right]^{+\pi/2}$, $v_y = \frac{A_o}{2k_o} \mathrm{Re}\left[e^{i(k_o x + k_o y)}\right]^{+\pi/2}$

[30%]

c) $\mathrm{Re}\left[e^{i(k_o x + k_o y)}\right] = \cos(k_o x + k_o y)$



[20%]

d) $\frac{\partial \omega_z}{\partial t} + \underbrace{\vec{v}.\nabla \omega_z}_{\substack{k_x v_x + k_o v_y \\ = 0 \text{ (continuity)}}} - \alpha x \frac{\partial \omega_z}{\partial x} = \omega_z \underbrace{\frac{\partial v_z}{\partial z}}_{\alpha}$

$\hookrightarrow \frac{dA}{dt} + Ai \frac{dk_x}{dt} x = \alpha A + \alpha x \, ik_x A$

$\rightarrow \frac{dA}{dt} = \alpha A$ and $\frac{dk_x}{dt} = k_x \alpha$

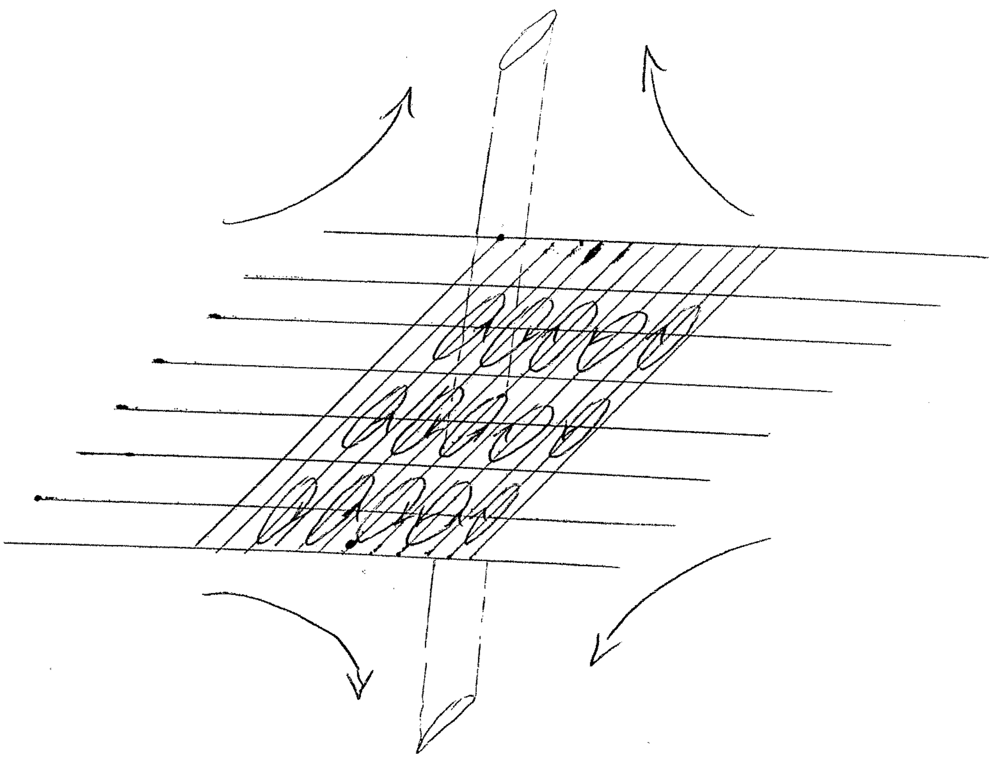
Hence $$A = A_0 e^{\alpha t}$$
$$k_x = k_0 e^{\alpha t}$$

Physical interpretation: the straining part of the flow $[-\alpha x, 0, \alpha z]$ squeezes the vortex tubes towards the plane $x=0$ and stretches this tubes along the z-direction:

hence A grows (exponentially)

and the width of the vortex tubes in the x-direction, πk_x^{-1} decreases (exponentially)

[30%]



Question 3

a) No shear stress on $\mathcal{C}$, hence conservation of the total angular momentum, hence $\int_S \omega \, dS$ is conserved [20%]

b) $\int_S \vec{u} \cdot [\text{Navier-Stokes}] \, dS$

$$= \int_S \frac{\partial}{\partial t}\left(\frac{u^2}{2}\right) + \underbrace{\vec{u}\cdot\vec{\nabla}\frac{u^2}{2}}_{0} + \underbrace{\vec{u}\cdot\vec{\nabla} p}_{0} - \nu\, \vec{u}\cdot\nabla^2\vec{u} \, dS = 0$$

$$\to \frac{\partial}{\partial t}\int_S \frac{u^2}{2} \, dS = -\nu \int_S \vec{\nabla}\vec{u} : \vec{\nabla}\vec{u} \, dS = -\nu \int_S \omega^2 \, dS \qquad [20\%]$$

c) $\int_S \omega \, [\text{vorticity equation}] \, dS$

$$= \int_S \frac{\partial}{\partial t}\left(\frac{\omega^2}{2}\right) + \underbrace{(\vec{u}\cdot\vec{\nabla})}_{0}\frac{\omega^2}{2} - \nu\, \omega \nabla^2 \omega \, dS = 0$$

$D = \int_S \omega^2 \, dS$, hence $\dfrac{\partial D}{\partial t} = -2\nu \int_S \vec{\nabla}\omega \cdot \vec{\nabla}\omega \, dS$

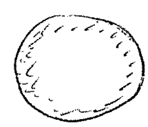
From lectures $\vec{\nabla}\omega$ is subjected to "stretching" $\to$ $\vec{\nabla}\omega$ can grow on a fast turnover timescale [20%]

d) Solid rotation: $u_\theta = \Omega r \to \omega = 2\Omega$

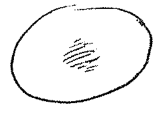
$$\left.\begin{aligned} \Phi_0 &= \int_S \omega \, dS = \int_0^R \omega \, 2\pi r \, dr = 2\pi R^2 \Omega \\ E_0 &= \int_S \frac{u_\theta^2}{2} \, dS = \int_0^R \frac{u_\theta^2}{2} 2\pi r \, dr = \frac{\pi}{4} R^4 \Omega^2 \end{aligned}\right\} \quad \Phi_0^2 = 16\pi E_0$$

Uniform vorticity distribution corresponds to the minimum $\int_S \omega^2 \, dS$ [20%]

e) When $\Phi_0^2 > 2\pi E_0$, we want less velocity with the same amount of vorticity $\longrightarrow$ concentrate vorticity on the boundary



When $\Phi_0^2 < 2\pi E_0$, vorticity is concentrated at the center [20%]



Question 4

a) $k = \frac{\overline{u^2}}{2}$. In the log region $u \sim U_*$

hence $k \sim U_*^2$ $\qquad k = a\, U_*^2$ [20%]

b) $\frac{U}{U_*} = A \ln y^+ + B$

$$= A \ln\left(\frac{y\, U_*}{\nu}\right) + B$$

At the first grid point $\boxed{\frac{U}{\sqrt{k/a}} = A \ln\left(\frac{y_0 \sqrt{k/a}}{\nu}\right) + B}$ [20%]

c) Energy dissipation $\varepsilon = \frac{u'^3}{\ell}$

In the log-law region, $\ell \sim y$, hence the energy dissipation is $\varepsilon = \frac{U_*^3}{y_0}$. [30%]

Or Energy production : Reynolds stress $\times$ mean velocity deformation

$$U_*^2 \frac{\partial U}{\partial y} \simeq U_*^2 \frac{U_*}{y_0}$$

$$\boxed{\varepsilon = A k^2 \frac{\sqrt{k/a}}{y_0}}$$

d) $\frac{\partial U}{\partial y}(y_0) = A \frac{U_*}{y_0} = A \frac{\sqrt{k/a}}{y_0}$

$$\boxed{\frac{\partial U}{\partial y}(y_0) = A \frac{\sqrt{k/a}}{y_0}}$$

[30%]

3 boundary conditions for the 3 unknowns U, ε, k