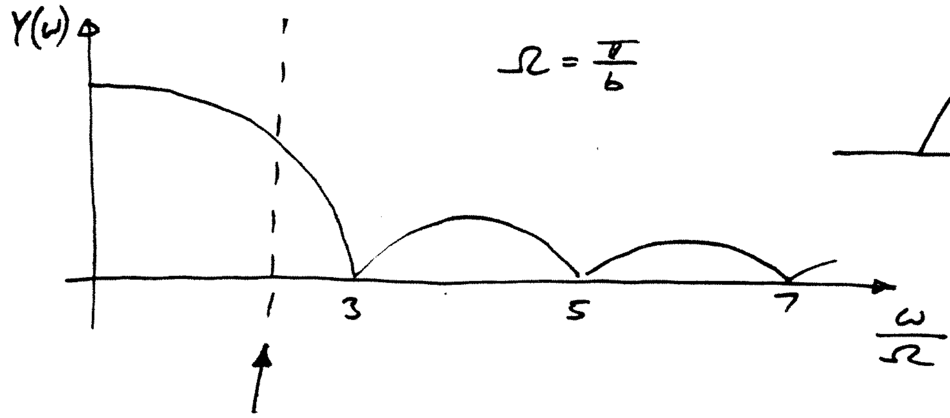
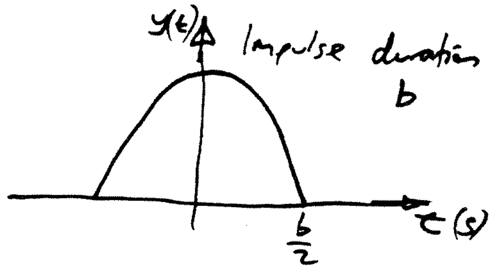
Module 4C6 2003 Solutions

1 (a)



$\Omega = \frac{\pi}{b}$



Useful frequency $\omega_{useful} \sim 2\Omega = \frac{2\pi}{b}$ (rad s⁻¹)

$\therefore \quad f_{useful} \sim \frac{1}{b}$ (Hz)

Say $f_{useful} = 100$ Hz $\therefore$ $\underline{\underline{b = 10\text{ ms}}}$

$\Omega = \frac{\pi}{b} \simeq 300 \text{ rad s}^{-1} = \sqrt{\frac{k}{m}}$

$\therefore \; k = m\Omega^2 = 1 \times (300)^2$

$= \underline{\underline{90000 \text{ N/m}}}$

[20%]

(b) Peak velocity = 10 ms^{-1}

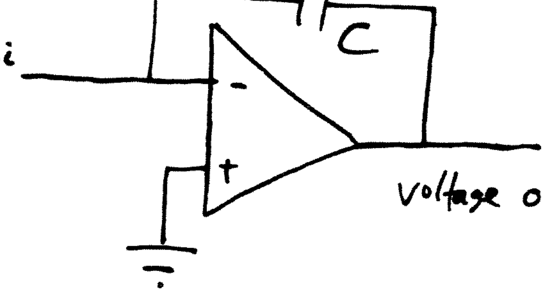
$\therefore$ Peak displacement $= \frac{v_{max}}{\Omega} = \frac{3}{300} = 0.01\text{m}$

Peak force $= k\,x_{max} = 900$ N

$\therefore$ Peak charge output $= 5 \times 900 = 4500$ pC

which corresponds to say 4.5V in the datalogger

$\therefore$ Charge amp gain $= 10^{-3}$ V/pC or 1 V/nC



$C = 1\text{nF}$

$V_o = \frac{Q_i}{C}$

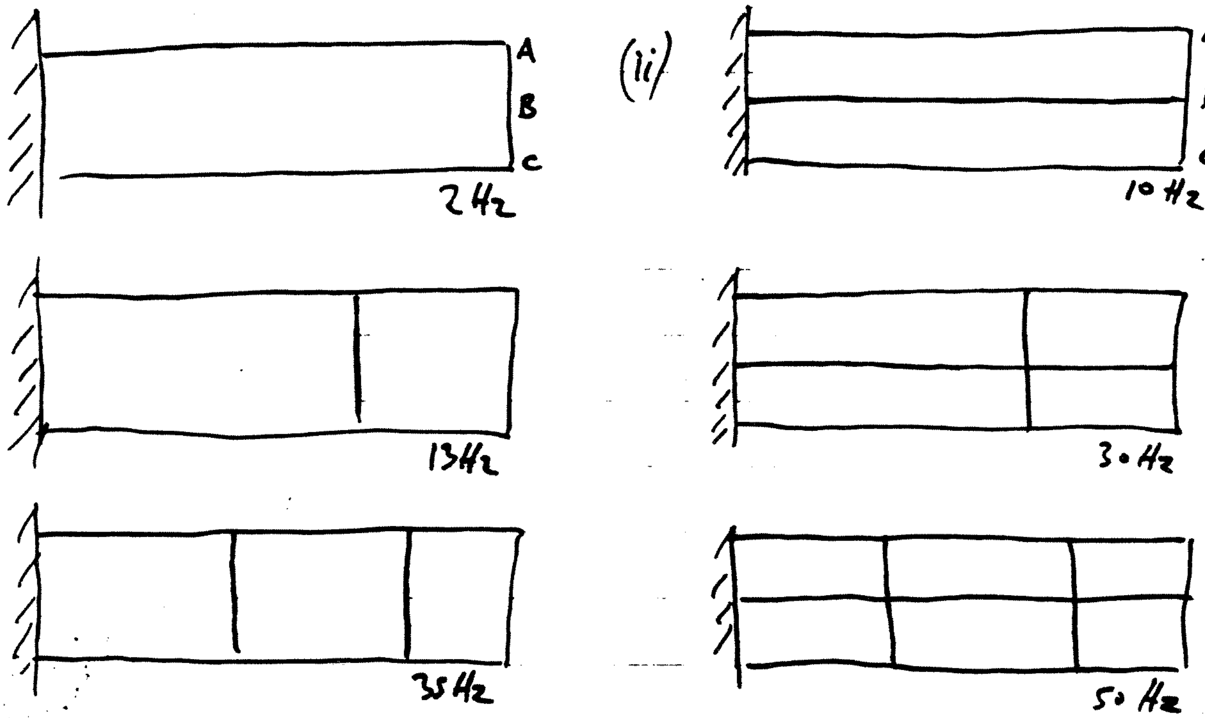
[20%]

1 cont.

(c) Want sampling rate well above any response frequency. The hammer excites usefully up to 100 Hz but there will be input measureable up to say 500 Hz. Nyquist criterion requires that we log at twice this

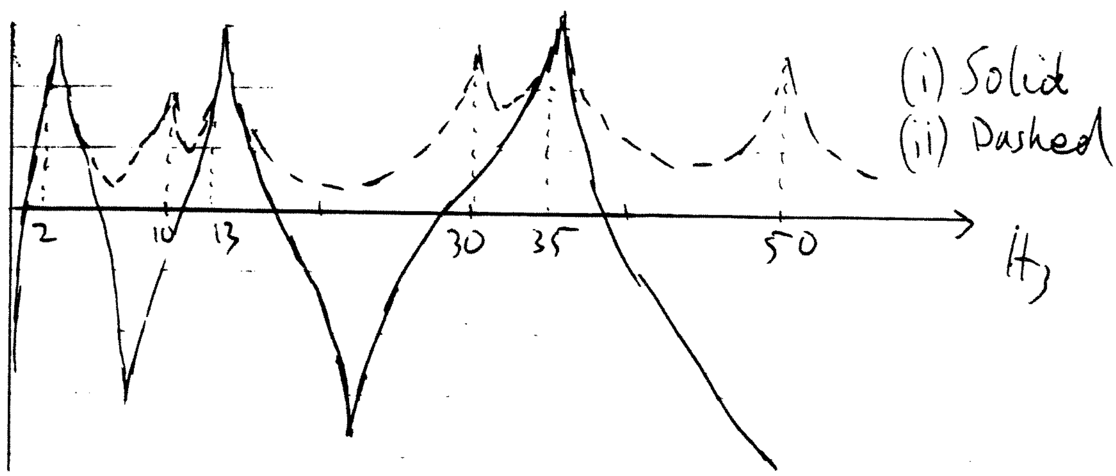
$\therefore$ $\underline{f_{sample} = 1000 \text{ Hz}}$ [20%]

(d) (i) (ii)



[20%]

(e)



Sign of $u_n(A)\,u_n(C)$ reverses for every mode pair: positive for bending, negative for torsion [20%]

2 (a) Note that system is symmetrical, so expect modes to be symmetric/antisymmetric.

So try (i) $\underline{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $K\underline{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2M\underline{u}$

so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a mode with $\omega^2 = 2$

(ii) $\underline{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$: then $K\underline{u} = \begin{bmatrix} 8 \\ -8 \end{bmatrix} = 8M\underline{u}$

$\therefore \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a mode with $\omega^2 = 8$

[30%]

(b) First-order form says $\dot{\underline{z}} = A\underline{z}$

with $\underline{z} = \begin{bmatrix} \underline{u} \\ \dot{\underline{u}} \end{bmatrix}$, $A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}$

So here, $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 3 & -0.3 & 0.1 \\ 3 & -5 & 0.1 & -0.3 \end{bmatrix}$

So (i) try $\underline{z} = \begin{bmatrix} 1 \\ 1 \\ \lambda \\ \lambda \end{bmatrix}$: $A\underline{z} = \begin{bmatrix} \lambda \\ \lambda \\ -2-0.2\lambda \\ -2-0.2\lambda \end{bmatrix}$

So $A\underline{z} = \lambda\underline{z}$ if $-2-0.2\lambda = \lambda^2$

i.e. $\lambda^2 + 0.2\lambda + 2 = 0$

$\therefore \lambda = \frac{1}{2}\left[-\frac{1}{5} \pm \sqrt{0.04-8}\right]$

$= i\omega(1+i\eta)$ with $\omega \simeq \sqrt{2}$ and $\eta = 0.1/\sqrt{2}$

(ii) try $\underline{z} = \begin{bmatrix} 1 \\ -1 \\ \lambda \\ -\lambda \end{bmatrix}$: $A\underline{z} = \begin{bmatrix} \lambda \\ -\lambda \\ -8-0.4\lambda \\ +8+0.4\lambda \end{bmatrix}$

So $A\underline{z} = \lambda\underline{z}$ if $\lambda^2 = -8-0.4\lambda$

$\therefore \lambda = \frac{1}{2}\left(-0.4 \pm \sqrt{0.16-32}\right) = i\omega(1+i\eta)$ with $\omega \simeq \sqrt{8}$, $\eta = \frac{0.2}{\sqrt{8}} = \frac{0}{0}$

[20%]

2 cont.

(c) $C^2 = 10^{-2}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} = 10^{-2}\begin{bmatrix} 10 & -6 \\ -6 & 10 \end{bmatrix} = \frac{K}{50}$ [10%]

(d) Let $C\underline{v}_n = \mu_n \underline{v}_n$ be eigenvector/eigenvalue of C

Then $C^2 \underline{v}_n = \mu_n (C\underline{v}_n) = \mu_n^2 \underline{v}_n$

$\therefore K\underline{v}_n = \alpha\mu_n^2 \underline{v}_n$

So $\underline{v}_n$ is an eigenvector of K with eigenvalue $\alpha\mu_n^2$.
So must have $\underline{v}_n = \underline{u}_n$, $\alpha\mu_n^2 = \omega_n^2$

where $\underline{u}_n$, ω_n are undamped modes and frequencies.
Now try $\underline{z} = \begin{bmatrix} \underline{u}_n \\ \lambda_n \underline{u}_n \end{bmatrix}$ as a solution of damped problem:

$$A\underline{z} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}\begin{bmatrix} \underline{u}_n \\ \lambda_n \underline{u}_n \end{bmatrix} = \begin{bmatrix} \lambda_n \underline{u}_n \\ -K\underline{u}_n - \lambda_n C\underline{u}_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_n \underline{u}_n \\ -\omega_n^2 \underline{u}_n - \lambda_n \mu_n \underline{u}_n \end{bmatrix}$$

So $A\underline{z} = \lambda_n \underline{z}$ if $\lambda_n^2 = -\omega_n^2 - \lambda_n \mu_n = -\omega_n^2 - \frac{\lambda_n \omega_n}{\sqrt{\alpha}}$

i.e. $\lambda_n^2 + \frac{\omega_n}{\sqrt{\alpha}}\lambda_n + \omega_n^2 = 0$

$$\therefore \lambda_n = \frac{\omega_n}{2}\underbrace{\left[-\frac{1}{\sqrt{\alpha}} \pm \sqrt{\frac{1}{\alpha} - 4}\right]}$$

Same complex factor for all modes, so all have the same loss factor η as seen in part (b)

[40%]

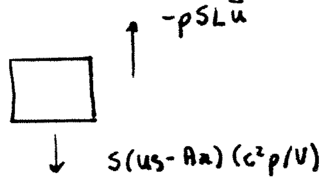
3 a) Consider displacements u and x:-

Change in volume $= uS - Ax$

Change (reduction) in density $= (uS-Ax)(\rho/V) = \Delta\rho$

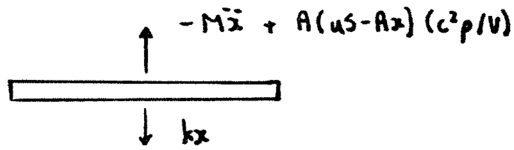
Reduction in pressure $= c^2 \Delta\rho = (uS-Ax)(c^2\rho/V)$

Consider force balance for the neck mass :-



$$\rho SL\ddot{u} + \left(\frac{\rho c^2S^2}{V}\right)u - \left(\frac{\rho c^2 SA}{V}\right)x = 0 \qquad (1)$$

Now consider the forces on the piston



$$M\ddot{x} + kx + \left(\frac{\rho c^2A^2}{V}\right)x - \left(\frac{\rho c^2 AS}{V}\right)u \qquad (2)$$

Writing equations (1) and (2) in matrix form:

$$\begin{pmatrix} \rho SL & 0 \\ 0 & M \end{pmatrix}\begin{pmatrix} \ddot{u} \\ \ddot{x} \end{pmatrix} + \begin{pmatrix} \frac{\rho c^2 S^2}{V} & -\frac{\rho c^2 SA}{V} \\ -\frac{\rho c^2 SA}{V} & k + \frac{\rho c^2 A^2}{V} \end{pmatrix}\begin{pmatrix} u \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

[3

b) When $u=0$, equation (2) reduces to $M\ddot{x} + kx + \left(\frac{\rho c^2A^2}{V}\right)x = 0$

$$\Rightarrow \underline{\omega_p^2 = \frac{kV + \rho c^2 A^2}{MV}}$$

When $x=0$, equation (1) reduces to $\rho SL\ddot{u} + \left(\frac{\rho c^2 S^2}{V}\right)u = 0$

$$\Rightarrow \underline{\omega_h^2 = \frac{c^2 S}{VL}}$$

[

3 cont.

c) Equations of motion can be written as:-

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \ddot{u} \\ \ddot{x} \end{pmatrix} + \begin{pmatrix} \omega_h^2 & -\alpha \\ -\beta & \omega_p^2 \end{pmatrix}\begin{pmatrix} u \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where α and β are positive quantities. Assuming simple harmonic motion:-

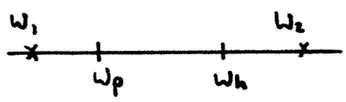
$$\begin{pmatrix} -\omega^2+\omega_h^2 & -\alpha \\ -\beta & -\omega^2+\omega_p^2 \end{pmatrix}\begin{pmatrix} u \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So: $(\omega^2-\omega_h^2)(\omega^2-\omega_p^2) - \alpha\beta = 0 \Rightarrow \omega^4 - \omega^2(\omega_h^2+\omega_p^2) + \omega_h^2\omega_p^2 - \alpha\beta = 0$

$$\omega^2 = \frac{\omega_h^2+\omega_p^2 \pm \sqrt{(\omega_h^2-\omega_p^2)^2+4\alpha\beta}}{2}$$

Assume $\omega_h > \omega_p$ (same argument if assume opposite) $\Rightarrow 2\omega^2 = \omega_h^2+\omega_p^2 \pm [(\omega_h^2-\omega_p^2) + \underset{\text{+ve number}}{\gamma}]$

$\Rightarrow \omega_1 < \omega_p \quad ; \quad \omega_2 > \omega_h$

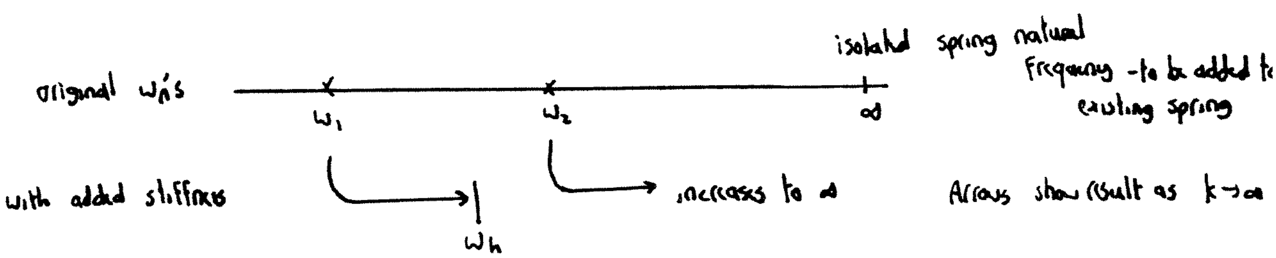


[30]

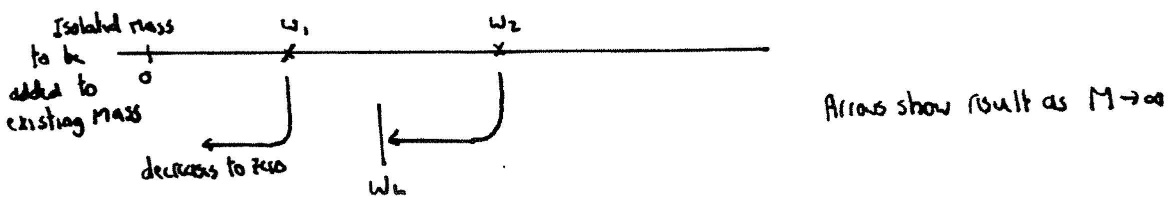
Agrees with interlacing: start with (ω_1, ω_2) and constrain u to give ω_p

or constrain x to give ω_h

d) Consider adding stiffness:-



Now consider adding mass



4 a) In polar coordinates :-

$$T\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) - \underbrace{m\frac{\partial^2 w}{\partial t^2}}_{+m\omega^2 w \text{ for harmonic vibration}} = 0$$

$w(r,\theta)$ = out of plane displacement

ω = vibration frequency.

The general solution is : $w(r,\theta) = A_1 J_n(kr)\cos n\theta + A_2 J_n(kr)\sin n\theta + A_3 Y_n(kr)\cos n\theta + A_4 Y_n(kr)\sin n\theta$

For $n = 0, 1, 2, 3 \ldots$

Where $J_n(kr)$ = Bessel function of the first kind of order n

$Y_n(kr)$ = Bessel function of the second kind of order n

k = wavenumber $= \omega\sqrt{\frac{m}{T}}$

[20]

b) Stress free boundary condition at $r=a$ ⇒ $\frac{\partial w}{\partial r} = 0$ at $r=a$

No singularity at $r=0$ ⇒ no contribution from $Y_n(kr)$

$$\Rightarrow \quad w(r,\theta) = A_1 J_n(kr)\cos n\theta + A_2 J_n(kr)\sin n\theta$$

$$\left.\frac{\partial w}{\partial r}\right|_{r=a} = 0 \quad \Rightarrow \quad \underline{J_n'(ka) = 0}$$

Thus to find the natural frequencies we need to find the points where $J_n'(z) = 0$. From the figure, these points are:-

$n=0 \Rightarrow z = 3.83$ ⟵ Mode 5

$n=1 \Rightarrow z = 1.84$ ⟵ Modes 1 and 2

$n=3 \Rightarrow z = 4.20$ ⟵ Modes 6 and 7

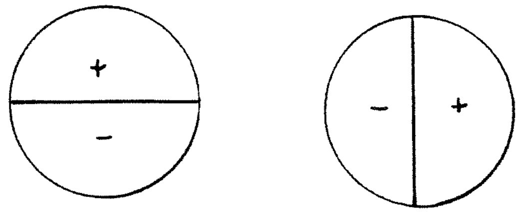
$n=2 \Rightarrow z = 3.05$ ⟵ Modes 3 and 4

} Mode numbering refers to non-zero modes.

Now $z = ka \Rightarrow z = \omega\sqrt{\frac{m}{T}}\,a \Rightarrow \omega = \left(\frac{z}{a}\right)\sqrt{\frac{m}{T}}^{-1} = 667z$

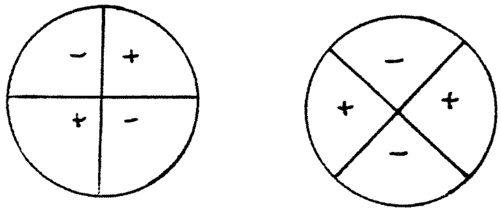
4 cont

Nodal lines:-



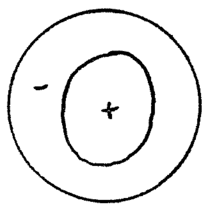
Modes 1 and 2

$\omega = 1227$ rad/s



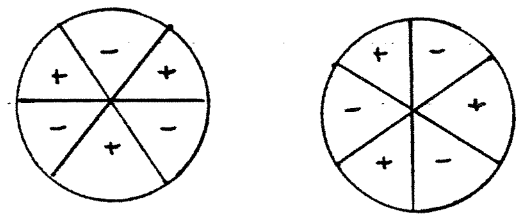
Modes 3 and 4

$\omega = 2033$ rad/s



Mode 5

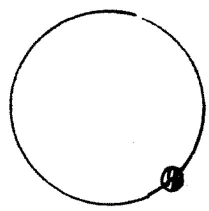
$\omega = 2553$ rad/s



Modes 6 and 7

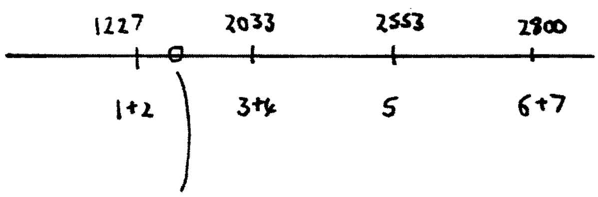
$\omega = 2800$ rad/s

c) Place the mass on an antinode of mode 3 and a node of mode 4, at the edge, when the displacement is maximum.



Mode 4 unchanged

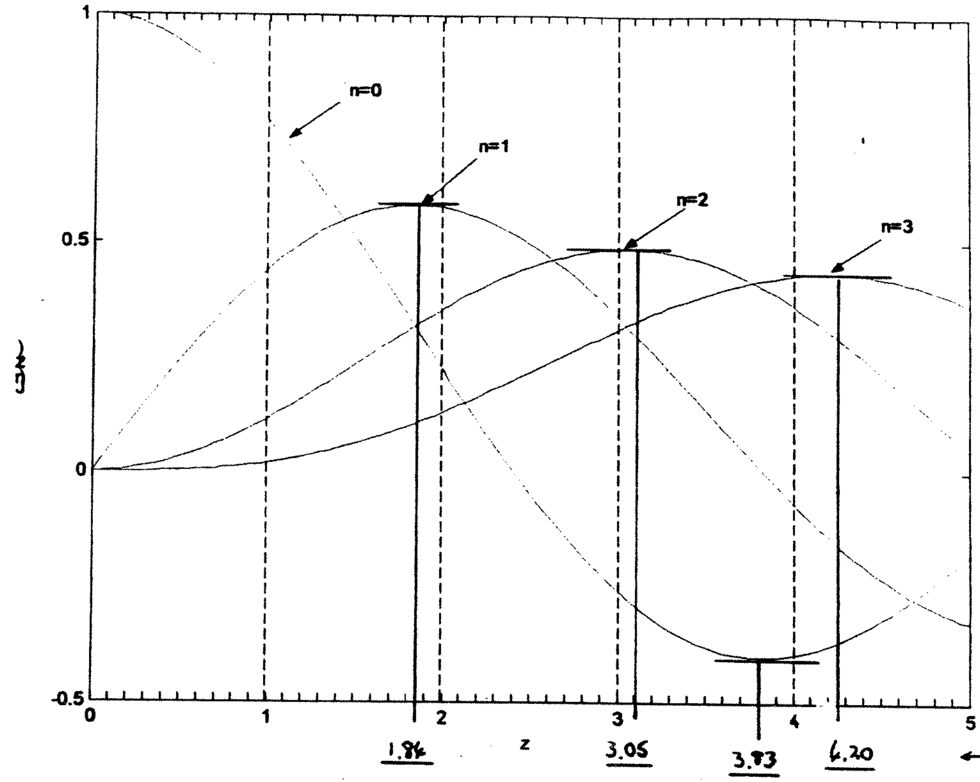
Mode 3 lowered



interlacing puts the new mode 3 in this region.

Maximum separation $= 2033 - 1227 = \underline{806 \text{ rad/s}}$

4 cont.



← Accurate values shown, rather than those deduced from graph!

Fig. Y The Bessel function $J_n(z)$ for various n