4C7 RANDOM AND NON-LINEAR VIBRATIONS - CRIBS 2003

1 a) $z(t) = x^2(t)$

$$R_{zz}(\tau) = E[z(t)\,z(t+\tau)] = E[x(t)x(t)\,x(t+\tau)\,x(t+\tau)]$$

Use formula (i) from question to give:-

$$R_{zz}(\tau) = E[x(t)x(t+\tau)]\,E[x(t)x(t+\tau)] + E[x(t)x(t+\tau)]\,E[x(t)x(t+\tau)] + E[x(t)x(t)]\,E[x^2(t+\tau)]$$

$$\underline{R_{zz}(\tau) = 2R_{xx}^2(\tau) + R_{xx}^2(0)}$$ [20%]

b) $$S_{zz}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{zz}(\tau)\,e^{-i\omega\tau}\,d\tau$$

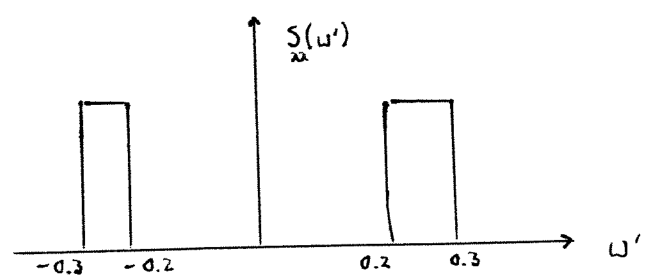
$$= \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} 2R_{xx}^2(\tau)\,e^{-i\omega\tau}\,d\tau}_{\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{xx}(\omega')\,S_{xx}(\omega'')\,e^{-i\omega'\tau}\,e^{+i\omega''\tau}\,e^{-i\omega\tau}\,d\omega'\,d\omega''\,d\tau} + \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} R_{xx}^2(0)\,e^{-i\omega\tau}\,d\tau}_{R_{xx}^2(0)\,\delta(\omega)}$$

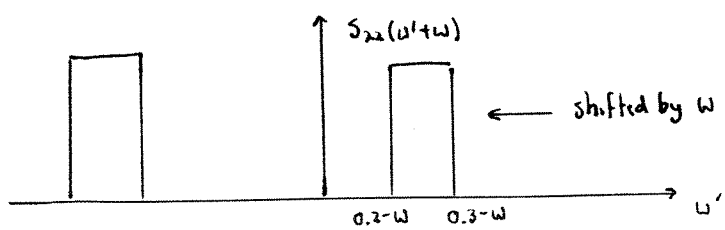
because $R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(\omega)\,e^{i\omega\tau}\,d\omega$

$$= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{xx}(\omega')\,S_{xx}(\omega'')\,\delta(\omega''-\omega'-\omega)\,d\omega'\,d\omega'' + R_{xx}^2(0)\,\delta(\omega)$$

$$\underline{S_{zz}(\omega) = 2\int_{-\infty}^{\infty} S_{xx}(\omega')\,S_{xx}(\omega'+\omega)\,d\omega' + R_{xx}^2(0)\,\delta(\omega)}$$ [35%]

c)



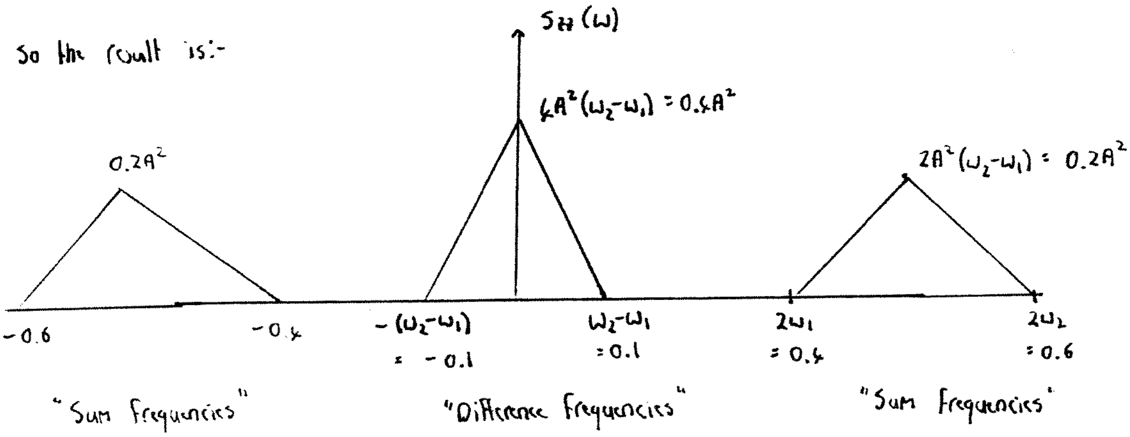


Product of functions $= A^2$ where blocks overlap.

Overlap region $= \omega_2 - \omega_1 - \omega$

etc

So the result is:-



[35%]

d) See above diagram - there is excitation at the resonant frequency. [10%]

2 a)

$$M\ddot{x}_1 + C\dot{x}_1 + kx_1 = F + \alpha G \qquad \text{— (1)}$$

$$M\ddot{x}_2 + C\dot{x}_2 + kx_2 = F - \alpha G \qquad \text{— (2)}$$

Since F and G are statistically independent, the spectum of $F + \alpha G$ is $S_{FF}(\omega) + \alpha^2 S_{GG}(\omega)$
$= (1+\alpha^2) S_0$

The spectrum of $F - \alpha G$ is also $(1+\alpha^2) S_0$ ⇒ Each oscillator has the same response spectrum

Using standard result for the response of a linear system to white noise:

$$\underline{\sigma_{x_1}^2 = \frac{\pi S_0 (1+\alpha^2)}{Ck} = \sigma_{x_2}^2} \qquad [25\%]$$

b) Subtract equations (1) and (2) to give:-

$$M(\ddot{x}_1 - \ddot{x}_2) + C(\dot{x}_1 - \dot{x}_2) + k(x_1 - x_2) = 2\alpha G$$

Putting $z = x_1 - x_2$ and using standard response to White Noise results:-

$$\underline{\sigma_z^2 = \frac{4\alpha^2 \cdot \pi S_0}{Ck}} \qquad \underline{\sigma_{\dot{z}}^2 = \frac{4\alpha^2 \pi S_0}{CM}} \qquad [20\%]$$

c)

$$\sigma_z^2 = \frac{4(0.5)^2 \pi \times 3 \times 10^{-5}}{0.04 \times 4} \Rightarrow \underline{\sigma_z = 0.024}$$

$$\sigma_{\dot{z}}^2 = \sigma_z^2 \left(\frac{k}{M}\right) \Rightarrow \underline{\sigma_{\dot{z}} = 0.048}$$

Impact rate = crossing rate ν_b^+ with $b = d = 0.1$

$$\nu_b^+ = \left(\frac{1}{2\pi}\right)\left(\frac{\sigma_{\dot{z}}}{\sigma_z}\right) e^{-\frac{1}{2}\left(\frac{0.1}{0.024}\right)^2} = \underline{5.4 \times 10^{-5}}$$

Prob of impact $= 1 - e^{-\nu_b^+ T}$; $T = 1 \times 60 \times 60$ ⇒ $\underline{\text{Prob} = 0.176}$ [30%]

d) In this case subtracting equations (1) and (2) gives:-

$$M(\ddot{x}_1-\ddot{x}_2)+C(\dot{x}_1-\dot{x}_2)+k(x_1-x_2) = \underbrace{(\gamma-1)F+(\gamma+1)\alpha G}$$

Input spectrum = $[(\gamma-1)^2+(\gamma+1)^2\alpha^2]S_0$

Put $\frac{\partial}{\partial\gamma}=0 \Rightarrow \quad 2(\gamma-1)+2(\gamma+1)\alpha^2=0$

$$\Rightarrow \quad \underline{\gamma=\frac{1-\alpha^2}{1+\alpha^2}}$$

[25%]

3 (a) $U = x^4 - 2x^2 + kx$, so governing equation is

$$\ddot{x} + 4x^3 - 4x + k = 0$$

If $k = 0$, write in standard first-order form as

$$\begin{cases} \dot{x} = y \\ \dot{y} = -4x^3 + 4x \end{cases}$$

Singular points where $y = 0$ and $4x^3 = 4x$
$\rightarrow x = 0, \pm 1$.

(i) Near $x = 0$ $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \simeq \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Eigenvalues λ satisfy $\begin{vmatrix} -\lambda & 1 \\ 4 & -\lambda \end{vmatrix} = 0$

$\therefore \lambda^2 = 4$, $\therefore \lambda = \pm 2$: saddle point

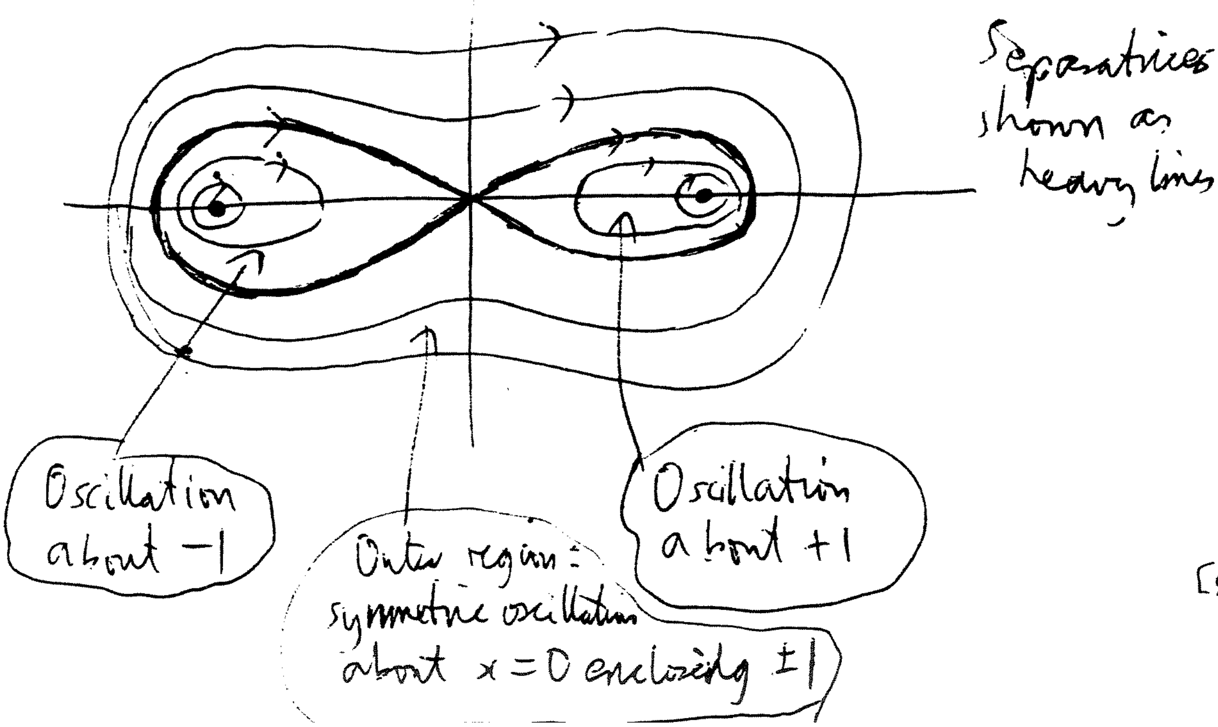
(ii) Near $x = 1$: let $x = 1 + \varepsilon$

Then $\begin{cases} \dot{\varepsilon} = y \\ \dot{y} = -4(1+\varepsilon)^3 + 4(1+\varepsilon) \simeq -8\varepsilon \end{cases}$

So eigenvalues λ satisfy $\begin{vmatrix} -\lambda & 1 \\ -8 & -\lambda \end{vmatrix} = 0$

$\therefore \lambda^2 = -8$, so $\lambda = \pm i\sqrt{8}$: centre

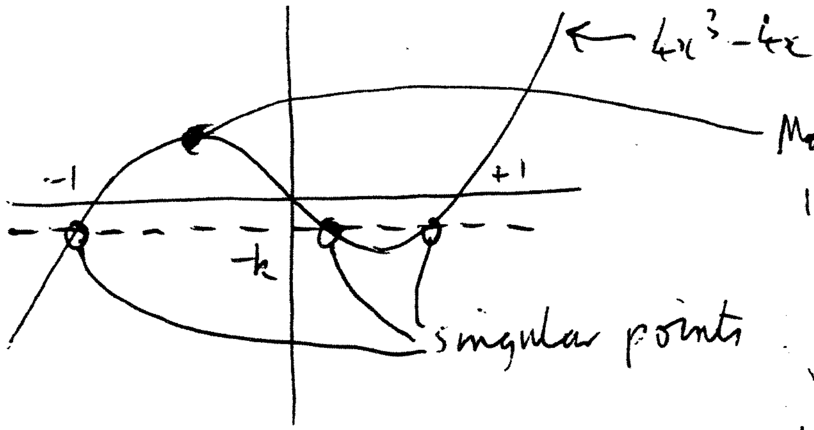
By symmetry, $x = -1$ is an identical centre.
So phase portrait is:



[50%]

(b) With $k \neq 0$, singular points satisfy $\begin{cases} y = 0 \\ -4x^3 + 4x - k = 0 \end{cases}$

ie where $4x^3 - 4x = -k$



Max value where $12x^2 - 4 = 0$

ie $x = \pm \frac{1}{\sqrt{3}}$

Value then is

$$\frac{\pm 4}{\sqrt{3}}\left(\frac{1}{3} - 1\right) = \frac{\pm 8}{3\sqrt{3}} = \pm k_{crit}$$

For $|k| < k_{crit}$, get 3 singular points with same character as part (a).

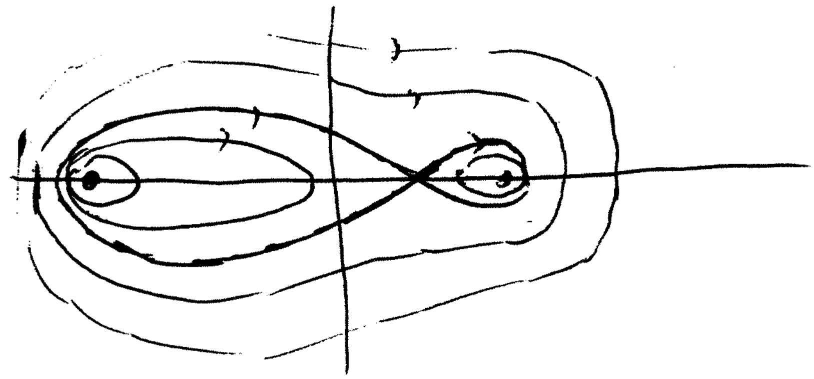
For $|k| > k_{crit}$, only one singular point which is a centre

[20%]

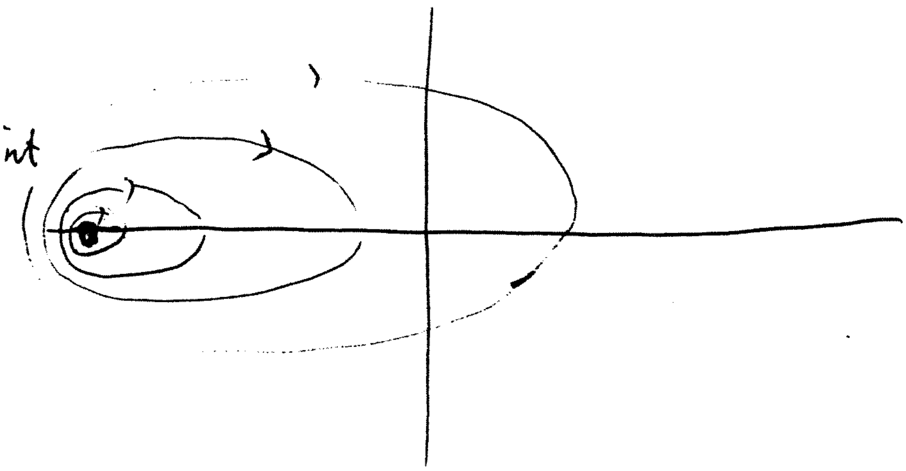
(c)

$|k| < k_{crit}$

$k > 0$

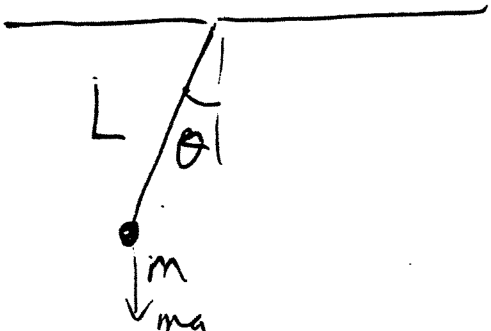


$k > k_{crit}$



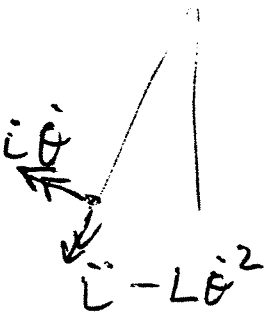
[40%]

4 (a)



$L = L_0 + a\sin\omega t$

Acceleration components are



To avoid needing to know tension in string, resolve perpendicular to string:

$$m(L\ddot\theta + 2\dot L\dot\theta) = -mg\sin\theta$$

$$\therefore\ L\ddot\theta + 2\dot L\dot\theta + g\sin\theta = 0$$

$$\therefore\ (L_0 + a\sin\omega t)\ddot\theta + 2a\omega\cos\omega t\,\dot\theta + g\sin\theta = 0$$ [25%]

(b) For $|\theta| << 1$, replace $\sin\theta \simeq \theta$

Then equation can be written

$$L_0\ddot\theta + g\theta \simeq -a\sin\omega t\,\ddot\theta - 2a\omega\cos\omega t\,\dot\theta$$

$$\therefore\ \ddot\theta + \Omega^2\theta \simeq -\frac{a}{L}\left[\ddot\theta\sin\omega t + 2\omega\dot\theta\cos\omega t\right]$$

$(\Omega^2 = g/L_0)$

With $\frac{a}{L} << 1$, solve by iteration.

Zero order: $\ddot\theta_0 + \Omega^2\theta_0 = 0$

$\rightarrow$ general solution $\theta_0 = A\cos(\Omega t + \phi)$, A, ϕ constant

First order: solve $\ddot\theta_1 + \Omega^2\theta_1 \simeq -\frac{a}{L}\left(\ddot\theta_0\sin\omega t + 2\omega\dot\theta_0\cos\omega t\right)$

$$\therefore \ddot{\theta}_1 + \Omega^2\theta_1 = \frac{Aa}{L}\left[\Omega^2\cos(\Omega t+\phi)\sin\omega t + 2\omega\Omega\sin(\Omega t+\phi)\cos\omega t\right]$$

$$= \frac{Aa}{L}\left\{\frac{\Omega^2}{2}\left[\sin(\omega t+\Omega t+\phi) + \sin(\omega t-\Omega t-\phi)\right]\right.$$
$$\left. + \omega\Omega\left[\sin(\omega t+\Omega t+\phi) - \sin(\omega t-\Omega t-\phi)\right]\right]$$

Provided $\omega+\Omega \neq \Omega$ and $\omega-\Omega \neq \Omega$, we can find a P.I. by trying $\theta_1 = \alpha\sin(\omega t+\Omega t+\phi) + \beta\sin(\omega t-\Omega t-\phi)$

Substitute:

$$-\alpha\left[(\omega+\Omega)^2-\Omega^2\right]\sin(\omega t+\Omega t+\phi) - \beta\left[(\omega-\Omega)^2-\Omega^2\right]\sin(\omega t-\Omega t-\phi)$$
$$= RHS$$

$$\therefore \alpha = -\frac{Aa\Omega}{2L}\frac{(\Omega+2\omega)}{(\omega^2+2\omega\Omega)}, \quad \beta = -\frac{Aa\Omega}{2L}\frac{(\Omega-2\omega)}{(\omega^2-2\omega\Omega)}$$

This solution describes steady, non-growing response.

$\omega+\Omega=\Omega$ is not possible, but $\omega-\Omega=\Omega \Rightarrow \omega=2\Omega$ is possible. Then for a P.I. have to try $\theta_1 = \alpha\sin(3\Omega t+\phi) + \gamma t\cos(\Omega t+\phi)$

α is the same as before.

For γ term:
$$\begin{cases} \dot{\theta}_1 = -\gamma t\Omega\sin(\Omega t+\phi) + \gamma\ldots\cos(\Omega t+\phi) \\ \ddot{\theta}_1 = -\gamma t\Omega^2\cos(\Omega t+\phi) - 2\gamma\Omega\sin(\Omega t+\phi)\end{cases}$$

$$\therefore \text{need } -2\gamma\Omega\sin(\Omega t+\phi) = \frac{Aa\Omega}{2L}(\Omega-2\omega)\sin(\Omega t+\phi)$$

$$\therefore \gamma = \frac{3Aa\Omega}{4L}$$ [50%]

(c) Growing oscillation only in the case $\omega = 2\Omega$. Phase ϕ plays no role – the value of γ does not depend on ϕ, rather surprisingly. [25%]