1. (a) See Lecture Notes for derivation of equations of motion (40%)

(b) (i) $\delta = -K\theta \qquad \Omega = \dot{\theta} \qquad$ ie $\bar{\Omega} = s\bar{\theta}$ (s = Laplace T/F)

So the eq's of motion are

$$m(\dot{v} + u\Omega) + (C_f + C_r)\frac{v}{u} + (aC_f - bC_r)\frac{\Omega}{u} + C_f\frac{K}{s}\Omega = 0$$

and

$$I\dot{\Omega} + (aC_f - bC_r)\frac{v}{u} + (a^2C_f + b^2C_r)\frac{\Omega}{u} + aC_f\frac{K}{s}\Omega = 0$$

putting $\bar{\dot{v}} = s\bar{v}$ & $\bar{\dot{\Omega}} = s\bar{\Omega}$ gives the characteristic eqn:

$$\begin{vmatrix} ms + \dfrac{C_f + C_r}{u} & mu + \left(\dfrac{aC_f - bC_r}{u}\right) + \dfrac{C_f K}{s} \\[2ex] \dfrac{aC_f - bC_r}{u} & Is + \left(\dfrac{a^2C_f + b^2C_r}{u}\right) + \dfrac{aC_f K}{s} \end{vmatrix} \begin{matrix} \bar{v} \\[2ex] \bar{\Omega} \end{matrix} = 0$$

Hence

$$\left[ms + \frac{C_f + C_r}{u}\right]\left[Is + \left(\frac{a^2C_f + b^2C_r}{u}\right) + \frac{aC_f K}{s}\right] - \left[\frac{aC_f - bC_r}{u}\right]\left[mu + \left(\frac{aC_f - bC_r}{u}\right) + \frac{C_f K}{s}\right] = 0$$

ie

$$\left[ms^2 + \left(\frac{C_f + C_r}{u}\right)s\right]\left[Is^2 + \left(\frac{a^2C_f + b^2C_r}{u}\right)s + aC_f K\right] - \left[\left(\frac{aC_f - bC_r}{u}\right)s\right]\left[mus + \left(\frac{aC_f - bC_r}{u}\right)s + C_f K\right] = 0$$

$$mIs^{\cancel{4}\,3} + \left[\frac{(a^2C_f + b^2C_r)m + (C_f + C_r)I}{u}\right]s^{\cancel{3}\,2}$$
$$+ \left[\frac{(C_f + C_r)(a^2C_f + b^2C_r)}{u^2} + maC_f K - \left(\frac{aC_f - bC_r}{u^2}\right)^2 - (aC_f - bC_r)m\right]s^{\cancel{2}\,1}$$
$$+ \left[\left(\frac{C_f + C_r}{u}\right)aC_f K - \left(\frac{aC_f - bC_r}{u}\right)C_f K\right]\cancel{s} = 0 \qquad (30\%)$$

Neglect $s = 0$ soln $\Rightarrow a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$

(ii) Use the Routh - Hurwitz Criteria in the data book

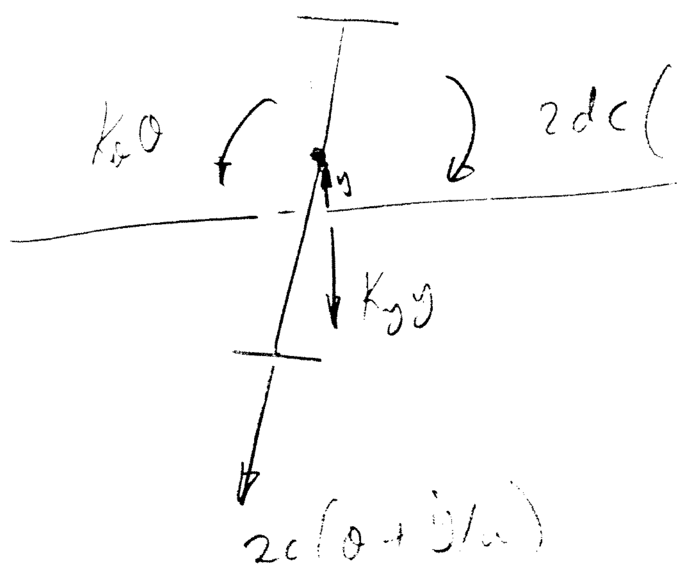
(i) All a's > 0

(ii) $a_1 a_2 > a_0 a_3$

Substitute for a's from Characteristic eqn. (10%)
→ obtain inequalities for K.

(iii) This strategy ensures that the vehicle remains parallel with the direction of motion ($\theta = 0$) but does not control the sideslip - so the lateral position can increase uncontrolled.
∴ Need another control loop with $-K'\frac{U}{s}$ (20%)

(2) (b) (a) See Lecture notes for derivation of ret. force & moment.



$$\boxed{\text{with } K_\theta = K_y a^2}$$

(50%)

$$\Sigma \underline{F} = 0: \quad 2c\left(\theta + \frac{\dot{y}}{u}\right) + K_y y = 0 \qquad (1)$$

$$\circlearrowright \Sigma M = 0: \quad 2d\, c\left(\frac{\varepsilon y}{r} - \frac{d\dot{\theta}}{u}\right) - K_\theta \theta = 0 \qquad (2)$$

$$(1) \rightarrow \theta = -\frac{K_y}{2c} y - \dot{y}/u$$

$$\& \quad \dot{\theta} = -\frac{K_y}{2c}\dot{y} - \ddot{y}/u \qquad (3)$$

(3) into (2) gives

$$\frac{\varepsilon y}{r} + \frac{d}{u}\left(-\frac{K_y}{2c}\dot{y} - \ddot{y}/u\right) - \frac{a^2 K_y}{2dc}\left(-\frac{K_y}{2c}y - \dot{y}/u\right) = 0$$

$$\ddot{y}\left(\frac{d}{u^2}\right) + \dot{y}\frac{K_y}{2dcu}\left(d^2 + a^2\right) + y\left(\frac{\varepsilon}{r} + \frac{a^2 K_y^2}{4dc^2}\right)$$

$$\text{i.e.} \quad \ddot{y} + \dot{y}\left[\frac{u K_y (d^2 + a^2)}{2d^2 c}\right] + y\left[\frac{u^2}{d}\left(\frac{\varepsilon}{r} + \frac{a^2 K_y^2}{4dc^2}\right)\right] = 0$$

(50%)

(c) Hunting wavelength

$$\omega_n^2 = \frac{U^2}{d}\left(\varepsilon/r + \frac{a^2 K_y^2}{4dc^2}\right)$$

$$\lambda = \frac{2\pi U}{\omega_n} = \frac{2\pi U}{U\sqrt{\frac{\varepsilon}{dr} + \frac{a^2 K_y^2}{4d^2c^2}}}$$

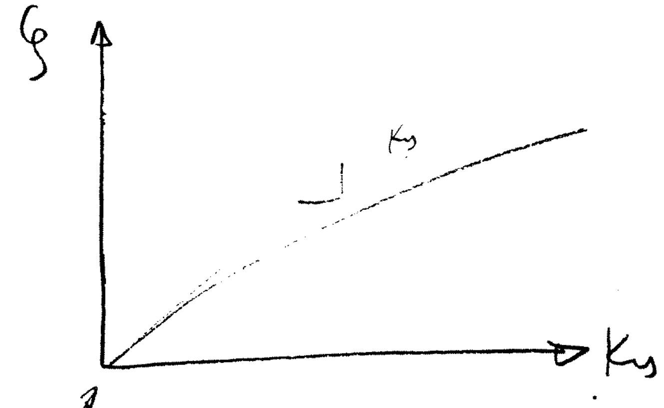
For zero suspension stiffness, $K_y = 0 \Rightarrow \lambda = 2\pi\sqrt{\frac{dr}{\varepsilon}}$ as per free wheelset

(10%)

(d) Damping ratio of hunting mode

$$\zeta = \frac{U K_y (d^2 + a^2)}{4d^2 C\sqrt{\frac{U^2}{d}\left(\frac{\varepsilon}{r} + \frac{a^2 K_y^2}{4dc^2}\right)}}$$

$$= \frac{1 + a^2/d^2}{4C}\left[\frac{K_y}{\sqrt{\frac{\varepsilon}{dr} + \frac{a^2 K_y^2}{4d^2c^2}}}\right]$$



zero for $K_y = 0$ as per free wheelset

(independent of speed)

(10%)

Q3

a) $E[(z_s - z_u)^2]$, the mean square suspension displacement. The space available for suspension movement is usually limited and so this response is often a constraint when selecting values for k and c.

$E[\ddot{z}_s^2]$, the mean square sprung mass acceleration is a measure of ride discomfort and the aim is usually to minimise this response.

$E[k_t(z_u - z_r)^2]$, the mean square dynamic tyre force is an indicator of road holding performance, or for heavy vehicles, an indicator of road damage. Again, the aim is usually to minimise this response.

The variable S_o is the spectral density of the white noise velocity input. A white noise velocity is a good approximation to the input provided by a random road profile in general. For a given road, S_o depends on the speed of the vehicle. (30%)

b)

$$E[\ddot{z}_s^2] = \frac{\pi S_o\left[(m_s + m_u)k^2 + k_t c^2\right]}{m_s^2 c}$$

$$\frac{\partial E[\ddot{z}_s^2]}{\partial c} = \frac{m_s^2 c\, \pi S_o 2 k_t c - \pi S_o\left[(m_s+m_u)k^2 + k_t c^2\right] m_s^2}{m_s^4 c^2}$$

equate to zero
$$2k_t c^2 = (m_s + m_u)k^2 + k_t c^2$$
$$k_t c^2 = (m_s + m_u)k^2$$
$$c = k\sqrt{\frac{m_s + m_u}{k_t}}$$

(15%)

c) $E\left[(k_t(z_r - z_u))^2\right] = \dfrac{\pi S_0}{M_s^2 c}\{A + (M_s + M_u)^2 k_t c^2\}$,

where $A = (M_s + M_u)^3 k^2 - 2k k_t M_u M_s (M_s - M_u) + M_u M_s^2 k_t^2$

$$\frac{dE[(k_t(z_r - z_u))^2]}{dc} = M_s^2 c \{\pi S_0 2 (M_s + M_u)^2 k_t c\} - \pi S_0 \{A + (M_s + M_u)^2 k_t c^2\} M_s^2$$

equate to zero

$$c^2 \, 2 (M_s + M_u)^2 k_t = A + (M_s + M_u)^2 k_t c^2$$

$$c^2 = \frac{A}{(M_s + M_u)^2 k_t}$$

$$c = \sqrt{\frac{(M_s + M_u)^3 k^2 - 2k k_t M_u M_s (M_s - M_u) + M_u M_s^2 k_t^2}{(M_s + M_u)^2 k_t}}$$ 25

d) Equate the expressions in (b) and (c) to give

$$\frac{(M_s + M_u)^3 k^2 - 2k k_t M_u M_s (M_s - M_u) + M_u M_s^2 k_t^2}{(M_s + M_u)^2 k_t} = k^2 \frac{(M_s + M_u)}{k_t}$$

$$-2k k_t M_u M_s (M_s + M_u) + M_u M_s \ldots = 0$$

$$\frac{2k}{k_t} = \frac{M_s}{M_s + M_u}$$

$$\therefore \quad \frac{1}{2}\frac{k_t}{k} = 1 + \frac{M_u}{M_s}$$

If this relationship is satisfied, an adjustable damper is not needed because acceleration and tyre force are minimised with one value of damping. The exception is of a higher value of damping is needed to constrain the suspension displacement.

Vehicles that do not come near to satisfying this relationship could benefit from an adjustable damper, for example to maximise roadholding in adverse conditions and to maximise comfort when it is safe to do so.

Typically for a car $\frac{m_u}{m_s} = \frac{1}{10}$ and $\frac{k_t}{k} = 7$,

so $\frac{1}{2} \cdot \frac{k_t}{k} = 7$ and $1 + \frac{m_u}{m_s} = 1{\cdot}1$, clearly not satisfying the relationship.

Some heavy vehicles have $\frac{m_u}{m_s} = \frac{1}{10}$ but $\frac{k_t}{k} = 2$,

so $\frac{1}{2} \frac{k_t}{k} = 1$ and $1 + \frac{m_u}{m_s} = 1{\cdot}1$, which nearly satisfies the relationship.

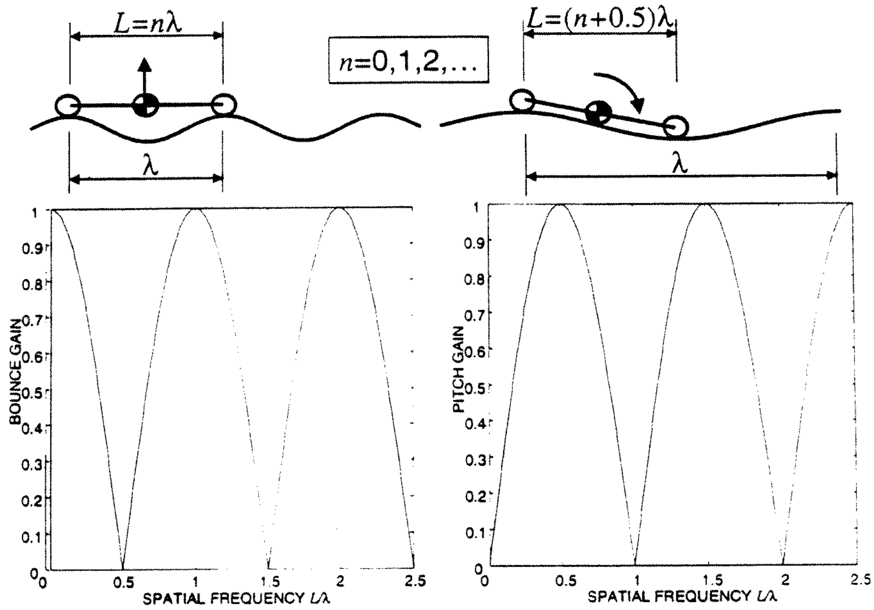
(Slightly deeper answers to the descriptive parts of the question might be expected)

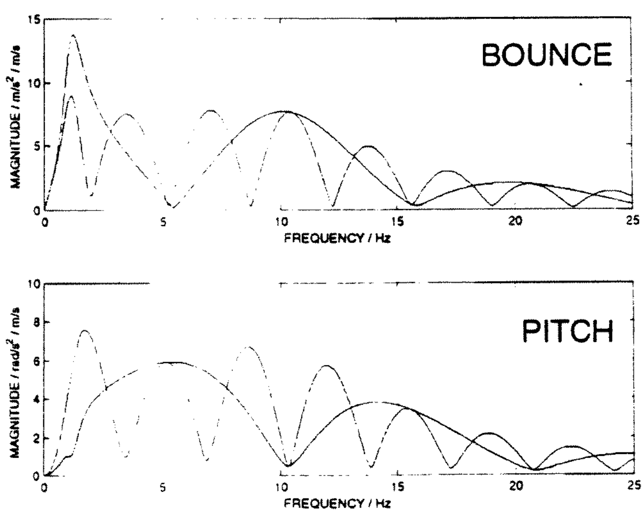
(30%)

Q4

a)

- Consider a massless, suspensionless vehicle, wheelbase L.
- The vehicle travels along a sinusoidal profile, wavelength λ.
- No pitch displacement when $L=n\lambda$, $n=0,1,2,\ldots$
- No bounce displacement when $L=(n+0.5)\lambda$, $n=0,1,2,\ldots$
- A random profile contains all wavelengths.
- The consequence is appearance of peaks and troughs and a dependence on speed.





Transfer functions of the four DOF pitch-plane model.
V=10m/s (dashed), V=30m/s (solid).

5/10

Q4

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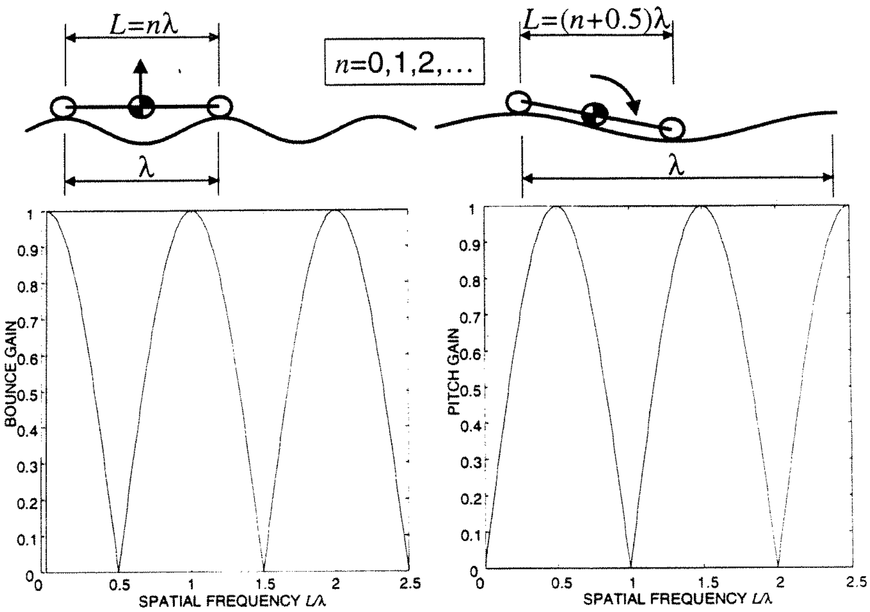
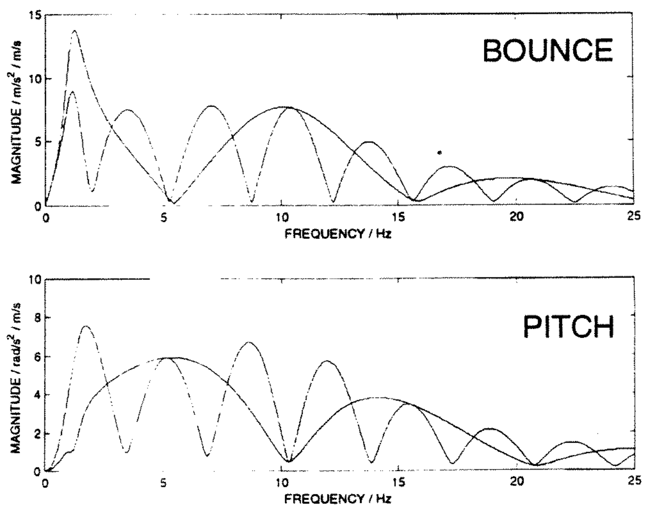
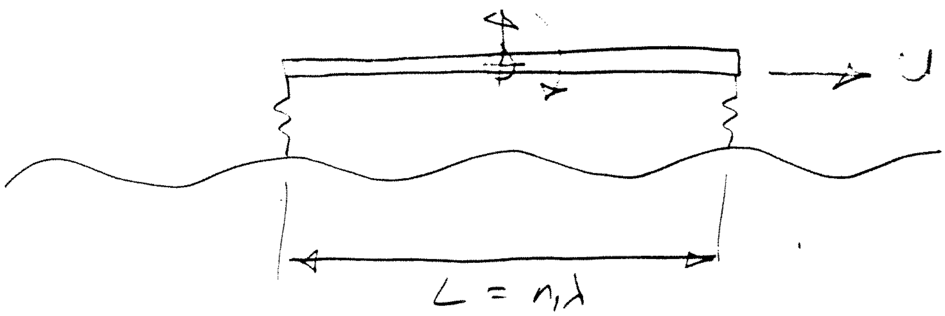




Transfer functions of the four DOF pitch-plane model.
V=10m/s (dashed), V=30m/s (solid).

b)



Pitch excitation minimised when $L = n_1\lambda$, n_1 is integer.
To minimise excitation at pitch mode natural frequency f_{pitch}, ensure:

$$U = f_{pitch}\,\lambda$$
$$U = f_{pitch}\,\frac{L}{n_1}$$

where $f_{pitch} = \frac{1}{2\pi}\sqrt{\frac{2ka^2}{I}}$

$$\therefore\ U = \frac{1}{2\pi}\sqrt{\frac{2ka^2}{I}}\,\frac{L}{n_1}$$

$$k = 2\left(\frac{\pi U n_1}{aL}\right)^2 I$$

$$\therefore\ k \propto U^2$$

What about n_1?
k is proportional to n_1^2, so choose $n_1 = 1$ to avoid high response to bounce excitation
thus $k = U^2 \cdot \frac{2I\pi^2}{a^2L^2}$ (25%)

c) Bounce excitation minimised when $L = (n_2 + \frac{1}{2})\lambda$
so need $U = f_{bounce} \cdot \frac{L}{(n_2 + \frac{1}{2})}$

where $f_{bounce} = \frac{1}{2\pi}\sqrt{\frac{2k}{m}}$

thus $k = 2m\,\frac{U^2\pi^2}{L^2}\left(n_2 + \frac{1}{2}\right)^2$

equate to expression for k in part (b):

$$\frac{U^2 2 I \pi^2}{a^2 L^2} = \frac{2m U^2 \pi^2 (n_z + \frac{1}{2})^2}{L^2}$$

$$\therefore I = ma^2 \left(n_z + \frac{1}{2}\right)^2$$

so to minimise bounce excitation at same time as pitch excitation, there are constraints on I.

If $n_z = 0$, $I = \frac{1}{4} ma^2$

If $n_z = 1$, $I = \frac{9}{4} ma^2$ etc.

Typical value of I for a car is $0.9 ma^2$, so these constraints on the value of I seem unrealistic.

(25%)