CRIB FOR PAPER 4C9

2002/3

① (a.) for unit square $\qquad$ symmetry $\Rightarrow d = 1$

$$\xi = \eta = 1 \longrightarrow 1 + 1 + \tfrac{1}{5}(1 - 6 + 1) = b \Rightarrow b = 6/5$$

[10%]

$$\xi = 1,\ \eta = 0 \longrightarrow 1 + \tfrac{1}{5}(1 - 0 + 0) = b \Rightarrow b = 6/5$$

(b) pot. function for square cross-section, $\phi = 0$ on bdy

$$\phi = C\left\{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 + \frac{1}{5}\left[\left(\frac{x}{a}\right)^4 - 6\left(\frac{x}{a}\right)^2\left(\frac{y}{a}\right)^2 + \left(\frac{y}{a}\right)^4\right] - 6/5\right\}$$

equil. requires $\nabla^2\phi = -2G\alpha = $ constant

$$\nabla^2\phi = \frac{2C}{a^2}\left\{1 + 1 + \frac{1}{5}\left[6\left(\frac{x}{a}\right)^2 - 6\left(\frac{x}{a}\right)^2 - 6\left(\frac{y}{a}\right)^2 + 6\left(\frac{y}{a}\right)^2\right]\right\} = \frac{4C}{a^2}$$

[25%]

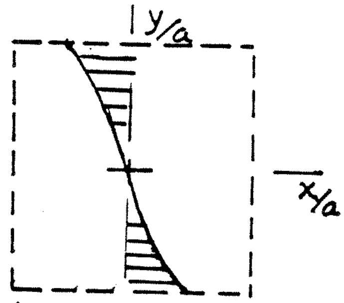
$$\therefore C = -\frac{G\alpha a^2}{2}$$

(c)

$$\sigma_{xz} = \frac{\partial\phi}{\partial y} = \frac{2C}{a}\left\{\frac{y}{a} + \frac{1}{5}\left[-6\left(\frac{x^2}{a^2}\right)\frac{y}{a} + 2\frac{y^3}{a^3}\right]\right\}$$

$$\sigma_{xz}(0,y) = \frac{2C}{a}y\left\{1 + \frac{2}{5}\frac{y^2}{a^2}\right\}$$

$\sigma_{xz}(0,y)$



[30%]

(d) torque

$$T = 2\int_{-a}^{a}\int_{-a}^{a}\phi(x,y)\,dx\,dy = 8a^2\int_0^1\int_0^1\phi(\xi,\eta)\,d\xi\,d\eta$$

$$= 8Ca^2\left\{\frac{1}{3} + \frac{1}{3} + \frac{1}{5}\left[\frac{1}{5} - \frac{2}{3} + \frac{1}{5}\right] - 6/5\right\} = -4.693\,C a^2$$

max stress $\qquad \sigma_{xz}(0,a) = -\dfrac{2T}{4.693\,a^3}\left\{1 + 2/5\right\} = -0.597\,T/a^3$

[35%]

Approx. shape is larger than square shaft so approximation gives increased stiffness and decreased stresses, $\quad 0.597\,T/a^3 < 0.622\,T/a^3$

2. (a) Drucker's postulates are only valid for stable materials: energy cannot be extracted from such materials in a closed cycle of applied stress.

From Drucker's postulates, one can show that: (i) plastic strain rate is perpendicular to the yield surface (Normality) (ii) the yield surface is convex (Convexity). Both are needed to establish an incremental ('flow') theory of plasticity.

[20%]

(b)

$$J_2 = \frac{1}{2} s_{ij} s_{ij} \quad , \quad s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

$$\dot{J}_2 = \frac{\partial J_2}{\partial t} = \frac{\partial J_2}{\partial \sigma_{ij}} \frac{\partial \sigma_{ij}}{\partial t}$$

But
$$\frac{\partial J_2}{\partial \sigma_{ij}} = s_{pq}\frac{\partial s_{pq}}{\partial \sigma_{ij}} = s_{pq}\left(\delta_{ip}\delta_{jq} - \frac{1}{3}\delta_{ij}\delta_{pq}\right)$$

$$= s_{pq}\delta_{ip}\delta_{jq} - \frac{1}{3}s_{pp}\delta_{ij} = s_{ij} = \frac{\partial J_2}{\partial s_{ij}}$$

For J_2 material with yield surface $J_2 - \text{const.} = 0$, the normal to y.s. is

$$n_{ij} = \frac{\partial J_2}{\partial s_{ij}} = s_{ij}$$

[25%]

$$\Rightarrow \quad \dot{J}_2 = s_{ij}\dot{\sigma}_{ij} = n_{ij}\dot{\sigma}_{ij} \, , \quad n_{ij} = s_{ij}$$

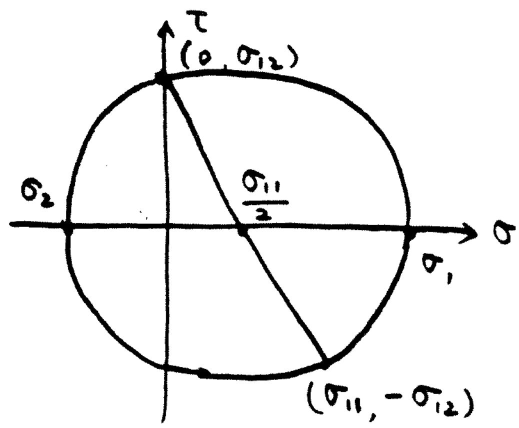
(c)

$$\sigma_1 = \frac{\sigma_{11}}{2} + \left(\frac{\sigma_{11}^2}{4} + \sigma_{12}^2\right)^{1/2}$$

$$\sigma_2 = \frac{\sigma_{11}}{2} - \left(\frac{\sigma_{11}^2}{4} + \sigma_{12}^2\right)^{1/2}$$

$$\sigma_3 = 0 \quad (\text{plane stress})$$

$$\Rightarrow \quad J_2 = \frac{1}{3}\left(\sigma_{11}^2 + 3\sigma_{12}^2\right)$$



$$\sigma_e = \sqrt{3J_2} = \sqrt{\sigma_{11}^2 + 3\sigma_{12}^2} \qquad \dot{\varepsilon}_{ij}^{pl} = \frac{1}{h} S_{ij} \dot{J}_2$$

$$\dot{J}_2 = \frac{2}{3}(\sigma_{11}\dot{\sigma}_{11} + 3\sigma_{12}\dot{\sigma}_{12}) \qquad \text{where } \frac{1}{h} = \frac{9}{4\sigma_e^2}\left(\frac{1}{E_T} - \frac{1}{E}\right)$$

J_2-flow theory:

$$\dot{\varepsilon}_{11} = \frac{\dot{\sigma}_{11}}{E} + \frac{1}{h} S_{11} \frac{2}{3}(\sigma_{11}\dot{\sigma}_{11} + 3\sigma_{12}\dot{\sigma}_{12})$$

$$\frac{1}{h} = \frac{9}{4\sigma_e^2}\left(\frac{1}{E_t} - \frac{1}{E}\right), \quad S_{11} = \begin{cases} 0 & \text{when } \sigma_{11} = 0 \\ \frac{2}{3}\sigma_{11} & \text{when } \sigma_{11} > 0 \end{cases}$$

Integration:

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} + \int_0^{\sigma_{11}} \frac{(2\sigma_{11}/3)^2 \, d\sigma_{11}}{\sigma_{11}^2 + 3\sigma_{12}^2} \cdot \frac{9}{4}\left(\frac{1}{E_t} - \frac{1}{E}\right)$$

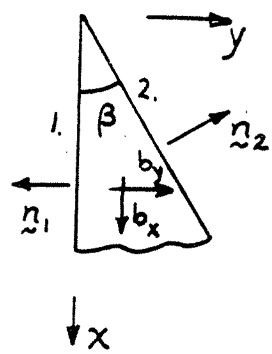
$$\rightarrow \sigma_{12} = \sigma_y / \sqrt{3}$$

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} + \left(\frac{1}{E_t} - \frac{1}{E}\right) \int_0^{\sigma_{11}} \frac{d\sigma_{11}}{1 + (\sigma_y/\sigma_{11})^2}$$

[45 %] $$= \frac{\sigma_{11}}{E} + \left(\frac{1}{E_t} - \frac{1}{E}\right)\left[\sigma_{11} - \sigma_y \tan^{-1}\left(\frac{\sigma_{11}}{\sigma_y}\right)\right]$$

(d) The answer to (c) will be different from above if J_2-deformation theory is used instead, because the loading path in Fig. 1 is not proportional. [10 %]

(3) (a)



from equilibrium

$$-b_x = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = -\alpha$$

$$-b_y = \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

$$\underset{\sim}{b} = \begin{Bmatrix} \alpha \\ 0 \end{Bmatrix}$$

traction on surfaces 1. & 2.

$$\underset{\sim}{t}_1 = \underline{\underline{\sigma}}(x,0)\cdot \underset{\sim}{n}_1 = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\underset{\sim}{t}_2 = \sigma(x, x\tan\beta)\cdot \underset{\sim}{n}_2 = \begin{bmatrix} \sigma_{xx}(x, x\tan\beta) & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} -\sin\beta \\ \cos\beta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

[25%] because $\sigma_{xx}(x, x\tan\beta) = \alpha(-x+x) = 0$

(b) Hooke's Law: $\varepsilon_{ij} = \frac{1}{E}\left\{(1+\nu)\sigma_{ij} - \nu\,\sigma_{kk}\delta_{ij}\right\}$

sub. definitions of deviatoric stress s_{ij} and strain e_{ij}

$$e_{ij} + \frac{e\delta_{ij}}{3} = \frac{1}{E}\left\{(1+\nu)(s_{ij} + \sigma_m\delta_{ij}) - 3\nu\,\sigma_m\delta_{ij}\right\}$$

$$e_{ij} + \cancel{\frac{3(1-2\nu)}{3E}\sigma_m\delta_{ij}} = \frac{1}{E}\left\{(1+\nu)\,s_{ij} + \cancel{(1-2\nu)\sigma_m\delta_{ij}}\right\}$$

[25%]

$$e_{ij} = \frac{(1+\nu)}{E}s_{ij} = \frac{1}{2G}s_{ij} \rightarrow s_{ij} = 2G\,e_{ij}$$

3 (b)

$$\dot{\varepsilon}^*_{rr} = \frac{\partial \dot{u}^*_r}{\partial r}, \quad \dot{\varepsilon}^*_{\theta\theta} = \frac{\dot{u}^*_r}{r}, \quad \dot{\varepsilon}^*_{zz} = 0$$

Where $\dot{u}^*_r = \dot{u}(r)$, $\dot{u}^*_\theta = \dot{u}^*_z = 0$ are kinematically admissible velocity field. Substituting into the incompressibility condition:

$$\dot{\varepsilon}^*_{rr} + \dot{\varepsilon}^*_{\theta\theta} + \dot{\varepsilon}^*_{zz} = \frac{d\dot{u}}{dr} + \frac{\dot{u}}{r} = 0$$

$\Rightarrow \quad u(r) = \dfrac{A}{r}$, A to be determined.

Upper bound theorem:

$$\int_{S_T} T_i^L \dot{u}_i^* dS \leq \int_V \sigma^*_{ij} \dot{\varepsilon}^*_{ij} dV$$

$$\int_{S_T} T_i^L \dot{u}_i^* ds = (2\pi t a)\, p^L \cdot \frac{A}{a} = 2\pi t p^L A$$

$$\int_V \sigma^*_{ij} \dot{\varepsilon}^*_{ij} dV = \int_V \sigma^*_e \dot{\varepsilon}^*_e dV$$

Where $\sigma^*_e = \sqrt{3}\tau_y$ (Mises) Data sheet 3C7

$$\dot{\varepsilon}^*_e = \sqrt{\frac{2}{3}\left(\dot{\varepsilon}^{*2}_{rr} + \dot{\varepsilon}^{*2}_{\theta\theta} + \dot{\varepsilon}^{*2}_{zz}\right)} = \frac{2A}{\sqrt{3}r^2}$$

$$\Rightarrow \int_V \sigma^*_e \dot{\varepsilon}^*_e dV = \sqrt{3}\tau_y \cdot 8 \int_0^{\pi/4} \int_a^{b/\cos\theta} \frac{2At}{\sqrt{3}r^2} \cdot r\,dr\,d\theta$$

$$= 16 A \tau_y t \int_0^{\pi/4} \ln\frac{b}{a\cos\theta}\, d\theta$$

$$\Rightarrow \quad 2\pi t p^L A \leq 16 A \tau_y t \left(\frac{\pi}{4}\ln\frac{b}{a} - \int_0^{\pi/4} \ln\cos\theta\, d\theta\right)$$

[50%] $\dfrac{b}{a} = 2 \Rightarrow p^L \leq 2\tau_y(2\ln 2 - 0.533) = 1.606\,\tau_y$