ENGINEERING TRIPOS PART IIB

Module 4C10 Examination, 2003

Answers

1. (a) $Q_B^{(2)} = Q_C^{(2)} = \dfrac{pL}{2}; \quad M_B^{(2)} = -M_C^{(2)} = \dfrac{pL^2}{12}$

(b) $K_r = \dfrac{2EI}{L^3}\begin{bmatrix} 4L^2 & L^2 \\ L^2 & 2L^2 \end{bmatrix}\begin{matrix}\theta_B \\ \theta_C\end{matrix}$; (c) $\begin{Bmatrix}\theta_B \\ \theta_C\end{Bmatrix} = \dfrac{pL^3}{24EI}\begin{Bmatrix} 3/7 \\ -5/7 \end{Bmatrix}$

(d) $u(x) = \theta_B\left[x - \dfrac{2x^2}{L} + \dfrac{x^3}{L^2}\right] + \theta_C\left[-\dfrac{x^2}{L} + \dfrac{x^3}{L^2}\right]; \quad 0 \le x \le L$

2. (a) $N_4 = 9\xi\eta$; (b) $\begin{Bmatrix} X \\ Y \end{Bmatrix} = L\begin{Bmatrix} 2\xi \\ \dfrac{\xi}{2} + \eta \end{Bmatrix}; \quad \left(L, \dfrac{3L}{4}\right)$

(c) $J = L\begin{bmatrix} 2 & 1/2 \\ 0 & 1 \end{bmatrix}$; (d) $|J| = 2L^2$; (e) $\xi_{xx} = \dfrac{u_2}{4L} + \dfrac{3u_4}{4L}$

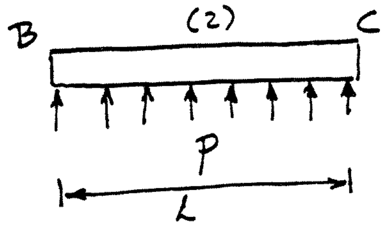
3 (c)(i) $\left\{\dfrac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 8L^2 \end{bmatrix} + \dfrac{P}{30L}\begin{bmatrix} 36 & -3L \\ -3L & 4L^2 \end{bmatrix}\right\}\begin{Bmatrix}\delta v_2 \\ \delta\theta_2\end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$; (ii) $P_{cr} = -\dfrac{20}{3}\dfrac{EI}{L^2}$

4 (b) $M_r = \dfrac{4m}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{matrix} u_2 \\ v_2 \end{matrix}$;

(c) $\omega_1^2 = 0.22\dfrac{k}{m}, \quad \dfrac{u_2}{v_2} = 0.414$; $\qquad \omega_2^2 = 1.28\dfrac{k}{m}, \quad \dfrac{u_2}{v_2} = -2.42$

PART IIB & EIST II
MODULE 4C10 FINITE ELEMENTS, 2003

① (a) equivalent loads



$$Q_B^{(2)} = \int_0^L p\, N(x)\, dx = \frac{pL}{2}$$

$$Q_C^{(2)} = \int_0^L p\, N_3(x)\, dx = \frac{pL}{2}$$

$$M_B^{(2)} = \int_0^L p\, N_2(x)\, dx = \frac{pL^2}{12} \qquad M_C^{(2)} = \int_0^L p\, N_4(x)\, dx = -\frac{pL^2}{12}$$ (15%)

(b) $u_2^{(1)} = u_1^{(2)}, \quad \theta_2^{(1)} = \theta_1^{(2)}$

$$K = \frac{2EI}{L^3}\begin{array}{c} \left[\begin{array}{cccccc} 6 & 3L & -6 & 3L & 0 & 0 \\ 3L & 2L^2 & -3L & L^2 & 0 & 0 \\ -6 & -3L & 6+6 & -3L+3L & -6 & 3L \\ 3L & L^2 & -3L+3L & 2L^2+2L^2 & -3L & L^2 \\ 0 & 0 & -6 & -3L & 6 & -3L \\ 0 & 0 & 3L & L^2 & -3L & 2L^2 \end{array}\right] \end{array} \begin{array}{c} u_A \\ \theta_A \\ u_B \\ \theta_B \\ u_C \\ \theta_C \end{array}$$

reduced stiffness matrix K_r from B.C. $u_A = u_B = u_C = \theta_A = 0$

$$K_r = \frac{2EI}{L^3}\begin{bmatrix} 4L^2 & L^2 \\ L^2 & 2L^2 \end{bmatrix} \begin{array}{c} \theta_B \\ \theta_C \end{array}$$ (30%)

(c) $K_r \underset{\sim}{d} = \underset{\sim}{R}$

$$\frac{2EI}{L}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \theta_B \\ \theta_C \end{Bmatrix} = \frac{pL^2}{12}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \rightarrow \begin{Bmatrix} \theta_B \\ \theta_C \end{Bmatrix} = \frac{pL^3}{24EI}\begin{Bmatrix} 3/7 \\ -5/7 \end{Bmatrix}$$ (3

(d) $\xi^{(1)} = x + L$

$u_A = u_B = \theta_A = 0 \rightarrow u(x) = \theta_B N_4(x) = \theta_B\left[-\frac{(x+L)^2}{L} + \frac{(x+L)^3}{L^2}\right], -L \le x \le 0$

$\xi^{(2)} = x$

$u_B = u_C = 0 \rightarrow u(x) = \theta_B N_2(x) + \theta_C N_4(x)$

$$= \theta_B\left[x - \frac{2x^2}{L} + \frac{x^3}{L^2}\right] + \theta_C\left[-\frac{x^2}{L} + \frac{x^3}{L^2}\right], \quad 0 \le x \le L$$

(25

(2)

(a)

N_4:

node	$N_4 = a + b\xi_j + c\eta_j + d\xi_j\eta_j$
$j=1$	$0 = a + 0 + 0 + 0$
$j=2$	$0 = a + b + 0 + 0$
$j=3$	$0 = a + 0 + c + 0$
$j=4$	$1 = a + b/3 + c/3 + d/9$

$$\left.\right\} \quad a=0,\ b=0,\ c=0,\ d=9$$

$N_1 = 1-\xi-\eta-3\xi\eta, \quad N_2 = \xi - 3\xi\eta, \quad N_3 = \eta - 3\xi\eta, \quad N_4 = 9\xi\eta$

(20%)

(b)

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ \vdots \end{Bmatrix}$$

$$= L \begin{bmatrix} 1-\xi-\eta-3\xi\eta & 0 & \xi-3\xi\eta & 0 & \eta-3\xi\eta & 0 & 9\xi\eta & 0 \\ 0 & 1-\xi-\eta-3\xi\eta & 0 & \xi-3\xi\eta & 0 & \eta-3\xi\eta & 0 & 9\xi\eta \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 2 \\ 1/2 \\ 0 \\ 1 \\ 2/3 \\ 1/2 \end{Bmatrix}$$

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = L \begin{Bmatrix} 2\xi \\ \xi/2 + \eta \end{Bmatrix}$$

for $\xi = \eta = 1/2$, $\quad x = L, \quad y = 3L/4$

i.e. edge 2-3 maps as straight line

(30%)

(c)

$$\underline{J} = \begin{bmatrix} \partial N_1/\partial\xi & \partial N_2/\partial\xi & \partial N_3/\partial\xi & \partial N_4/\partial\xi \\ \partial N_1/\partial\eta & \partial N_2/\partial\eta & \partial N_3/\partial\eta & \partial N_4/\partial\eta \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

$$= L \begin{bmatrix} -1-3\eta & 1-3\eta & -3\eta & 9\eta \\ -1-3\xi & -3\xi & 1-3\xi & 9\xi \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 1/2 \\ 0 & 1 \\ 2/3 & 1/2 \end{bmatrix} = L \underbrace{\begin{bmatrix} 2 & 1/2 \\ 0 & 1 \end{bmatrix}}$$ (20%)

(d) $|\underline{J}| = 2L$; i.e. area of global element
= 2 x area parent element (10%)

(e)

$$\underline{J}^{-1} = \begin{bmatrix} \partial\xi/\partial x & \partial\eta/\partial x \\ \partial\xi/\partial y & \partial\eta/\partial y \end{bmatrix} = \frac{1}{L} \begin{bmatrix} 1/2 & -1/4 \\ 0 & 1 \end{bmatrix}$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = J^{-1}_{11} \frac{\partial u}{\partial \xi} + J^{-1}_{12} \frac{\partial u}{\partial \eta} = J^{-1}_{11} \sum \frac{\partial N_i}{\partial \xi} u_i + J^{-1}_{12} \sum \frac{\partial N_i}{\partial \eta} u_i$$

with $u_1 = u_3 = 0$

$$\varepsilon_{xx} = \frac{1}{2L}\left[(1-3\eta)u_2 + 9\eta u_4\right] - \frac{1}{4L}\left[-3\xi u_2 + 9\xi u_4\right]$$

at centroid $\xi = \eta = 1/3$

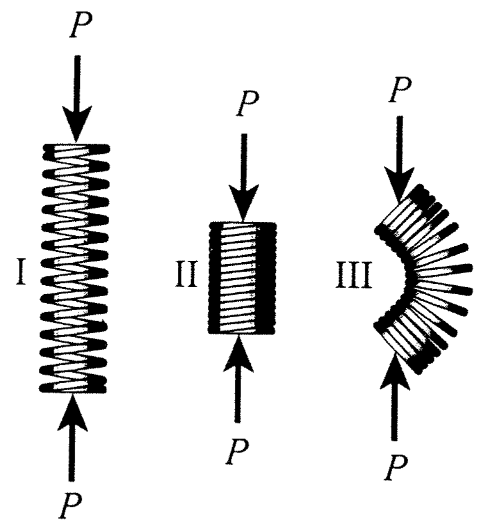
$$\varepsilon_{xx} = \frac{u_2}{4L} + \frac{3u_4}{4L}$$ (20%)

~~2~~. 3 (a) Geometric (change of geometry important);

Material (constitutive laws of flowing metal highly nonlinear);

Displacement boundary conditions (contact conditions). (20%)

(b)



In state I, the spring behaves as a linear spring. In state II, the spring has "bottomed out" and is much stiffer. In state III, the spring has buckled as an Euler column and its load capacity rapidly diminishes. (20%)

(c) (i)

Element II

$$[k'] = \frac{EI}{L^3} \begin{array}{cc} \begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ \text{sym.} & & & 4L^2 \end{bmatrix} & \begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array} \end{array}$$

$u_1 = u_2 = 0$ (rigid beam)

$$[k'_\sigma] = \frac{P}{30L} \begin{array}{cc} \begin{bmatrix} 36 & 3L & -36 & 3L \\ & 4L^2 & -3L & -L^2 \\ & & 36 & -3L \\ \text{sym} & & & 4L^2 \end{bmatrix} & \begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array} \end{array}$$

(due to stress stiffening)

~~2~~ 3. (continued)

Element ~~D~~

(global coordinates)

$$[k^2] = \frac{EI}{L^3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ & 4L^2 & 0 & 2L^2 \\ & \text{sym.} & 0 & 0 \\ & & & 4L^2 \end{bmatrix} \begin{matrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{matrix}$$

$[k^2_\sigma]$ vanishes because there is no axial force on ~~D~~.

Assemble together to get (noting that $v_3 = \theta_3 = v_1 = \theta_1 = 0$)

$$\Big\{ \underbrace{\frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 8L^2 \end{bmatrix}}_{K} + \underbrace{\frac{P}{30L}\begin{bmatrix} 36 & -3L \\ -3L & 4L^2 \end{bmatrix}}_{K_\sigma} \Big\} \begin{Bmatrix} \delta v_2 \\ \delta\theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (35\%)$$

(ii) At a critical load P_{cr} the structure buckles, with

$$\det\left[K + K_\sigma \right] = 0$$

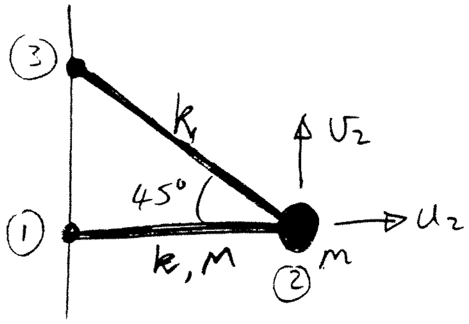
$$\Rightarrow \begin{vmatrix} 12(1+3\lambda) & -3L(2+\lambda) \\ -3L(2+\lambda) & 4L^2(2+\lambda) \end{vmatrix} = 0 \qquad \left(\lambda = \frac{PL^2}{30EI}\right)$$

The lowest eigenvalue is $\lambda = -\,2/9$

$$\Rightarrow \boxed{P_{cr} = -\,\frac{20}{3}\,\frac{EI}{L^2}}$$

(25%)

ii.



(a) Stiffness matrix

$$[k_{12}] = k\begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_2 \\ v_2 \end{array}$$

(c = 1, s = 0
Cook p79)

$$[k_{23}] = \frac{k}{2}\left[\begin{array}{cc|cc} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ \hline -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right]\begin{array}{c} u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}$$

$\phi = 135°$
$c = -1/\sqrt{2}$
$s = 1/\sqrt{2}$

$$\text{so } [K] = \frac{k}{2}\begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{array} \\ \left[\begin{array}{cc|cc|cc} 2 & 0 & -2 & 0 & & \\ 0 & 0 & 0 & 0 & \multicolumn{2}{c}{0} \\ \hline -2 & 0 & 3 & -1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ \hline & & -1 & 1 & 1 & -1 \\ \multicolumn{2}{c|}{0} & 1 & -1 & -1 & 1 \end{array}\right] \end{array}\begin{array}{c} \\ u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}$$

(30%)

(b) Mass matrix

$$[M_{12}] = m\begin{array}{c} \begin{array}{cccc} \ddot{u}_1 & \ddot{v}_1 & \ddot{u}_2 & \ddot{v}_2 \end{array} \\ \left[\begin{array}{cc|cc} 1/3 & 0 & 1/6 & 0 \\ 0 & 1/3 & 0 & 1/6 \\ \hline 1/6 & 0 & 1/3 & 0 \\ 0 & 1/6 & 0 & 1/3 \end{array}\right] \end{array}\begin{array}{c} \\ \ddot{u}_1 \\ \ddot{v}_1 \\ \ddot{u}_2 \\ \ddot{v}_2 \end{array}$$

(Cook p231)

$$[M_{lumped}] = m\left[\begin{array}{c|cc} 0 & \multicolumn{2}{c}{0} \\ \hline 0 & 1 & 0 \\ & 0 & 1 \end{array}\right]\begin{array}{c} \ddot{u}_1 \\ \ddot{v}_1 \\ \ddot{u}_2 \\ \ddot{v}_2 \end{array}$$

$$\text{so } [M] = m\left[\begin{array}{cc|cc} 1/3 & 0 & 1/6 & 0 \\ 0 & 1/3 & 0 & 1/6 \\ \hline 1/6 & 0 & 4/3 & 0 \\ 0 & 1/6 & 0 & 4/3 \end{array}\right]\begin{array}{c} \ddot{u}_1 \\ \ddot{v}_1 \\ \ddot{u}_2 \\ \ddot{v}_2 \end{array}$$

(20%)

(c) Reduced equation of motion for free vibration.

$$\frac{4m}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{u}_2 \\ \ddot{v}_2 \end{Bmatrix} + \frac{k}{2}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \underset{\sim}{0}$$

The eigenvalue problem is $([K] - \omega^2[M])\underset{\sim}{u} = 0$

$$\Rightarrow \begin{vmatrix} 3k/2 - \omega^2 4m/3 & -k/2 \\ -k/2 & k/2 - \omega^2 4m/3 \end{vmatrix} = 0$$

(c). cont

$$\left(\frac{3k}{2} - \omega^2 \frac{4m}{3}\right)\left(\frac{k}{2} - \omega^2 \frac{4m}{3}\right) - \frac{k^2}{4} = 0$$

$$\frac{k^2}{2} - \omega^2\left(\frac{2mk}{3} + 2mk\right) + \omega^4 \frac{16m^2}{9} = 0$$

$$\omega^4 - \frac{\frac{8}{3}mk}{\frac{16}{9}m^2} + \frac{\frac{k^2}{2}}{\frac{16m^2}{9}} = 0$$

$\frac{8}{3} \cdot \frac{9}{16}\frac{3}{2}$

i.e. $$\omega^4 - \frac{3}{2}\frac{k}{m} + \frac{9}{32}\frac{k^2}{m^2} = 0$$

$$\therefore \omega^2 = \frac{+\frac{3}{2}\frac{k}{m} \pm \sqrt{\left(\frac{3}{2}\frac{k}{m}\right)^2 - 4\left(\frac{9}{32}\right)\frac{k^2}{m^2}}}{2}$$

$$= \frac{k}{m}\left(\frac{3}{4} \pm \frac{3}{4\sqrt{2}}\right) = \frac{3}{4}\frac{k}{m}\left(1 \pm \frac{1}{\sqrt{2}}\right)$$

$$\therefore \underline{\underline{\omega^2 = 0.22\,k/m,\ 1.28\,k/m}}$$

<u>e - vectors</u>

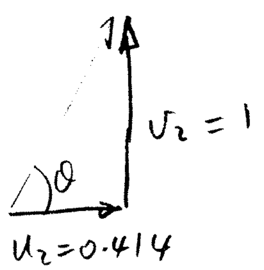
1st row of eqn of motion is

$$\left(\frac{3k}{2} - \omega^2 \frac{4m}{3}\right)u_2 - \frac{k}{2}v_2 = 0$$

i.e. $$\frac{u_2}{v_2} = \frac{k/2}{3k/2 - \omega^2\, 4m/3}$$

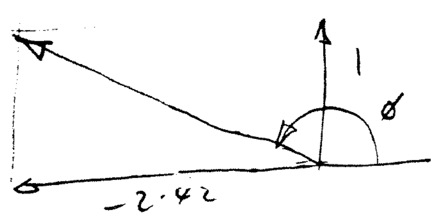
For $\omega_1^2 = 0.22\,k/m$, this gives $\frac{u_2}{v_2} = \frac{1/2}{3/2 - 4/3(0.22)} = 0.414$

For $\omega_2^2 = 1.28\,k/m$ $\qquad \frac{u_2}{v_2} = \frac{1/2}{3/2 - \frac{4}{3}(1.28)} = -2.42$



$$\theta = \tan^{-1}\frac{1}{0.414} = 67.5^\circ$$

@ $\omega_1^2 = 0.22\,k/m$



$$\phi = 180 - \tan^{-1}\frac{1}{2.42} = 157.5^\circ.$$

@ $\omega_2^2 = 1.28\,k/m$

The two modes are geometrically orthogon