2003 Tripos 4C12

1 (a) $\omega^2 = A k^\lambda \rightarrow \omega = A^{1/2} k^{\lambda/2}$

Phase velocity $c_p = \frac{\omega}{k} = A^{1/2} k^{\lambda/2 - 1}$

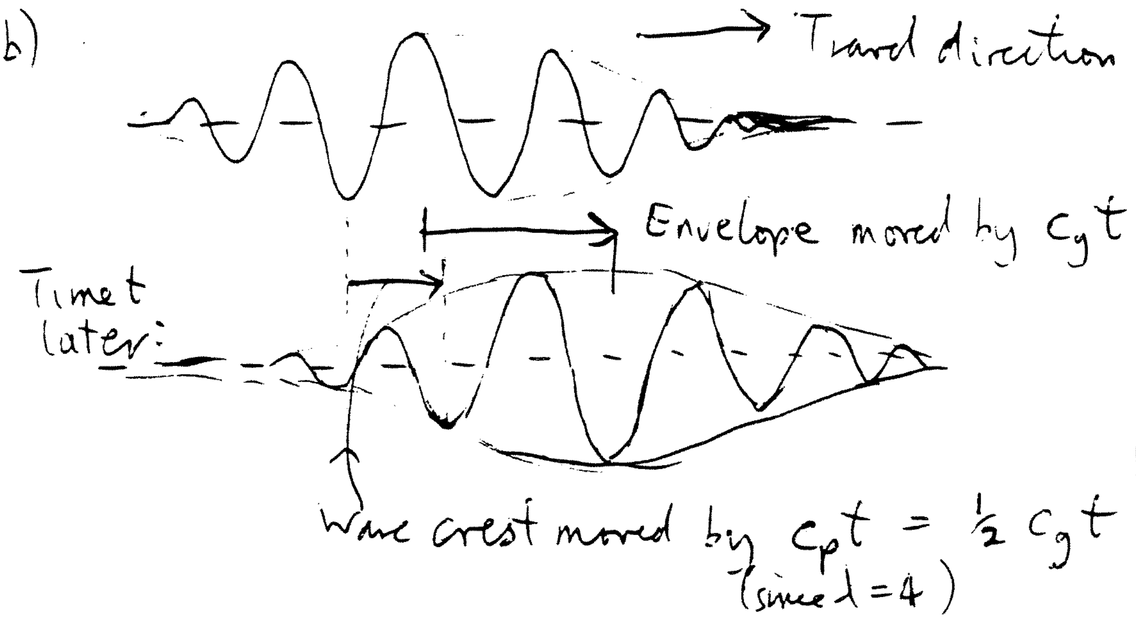
Group velocity $c_g = \frac{d\omega}{dk} = \frac{\lambda}{2} A^{1/2} k^{\lambda/2 - 1} = \frac{\lambda}{2} c_p$

deep Water waves have $\lambda = 1$

Compressional waves in a rod have $\lambda = 2$ [20%]

Bending waves in an Euler beam have $\lambda = 4$

(b)



[20%]

(c) Substitute $w = e^{i(k_1 x + k_2 y - \omega t)}$

$\rightarrow \omega^2 = D_1 k_1^4 + D_2 k_1^2 k_2^2 + D_3 k_2^4$ 1

At a fixed time t, w is constant along lines such that $k_1 x + k_2 y = \text{constant}$

$\rightarrow y = -\frac{k_1}{k_2} x + \frac{\text{constant}}{k_2}$

So slope of wave crests $= -\frac{k_1}{k_2} = \tan\alpha$ where

α is angle to x axis

Waves are travelling perpendicular to this, at angle θ such that $(k_1, k_2) = k(\cos\theta, \sin\theta)$

so $k = \sqrt{k_1^2 + k_2^2}$.

So phase velocity vector is $\underset{\sim}{c}_p = \underbrace{\frac{\omega}{k}}\,(\cos\theta, \sin\theta)$ [20%]

because k is wavenumber of one-dimensional waves along line of travel.

Group velocity $\underset{\sim}{c}_g = \left[\frac{\partial\omega}{\partial k_1}, \frac{\partial\omega}{\partial k_2}\right]$

So from (1), $2\omega\,\underset{\sim}{c}_g = \left[4D_1k_1^3 + 2D_2k_1k_2^2,\ 4D_3k_2^3 + 2D_2k_1^2k_2\right]$

$$= 2k^3\left[2D_1\cos^3\theta + D_2\sin^2\theta\cos\theta,\ 2D_3\sin^3\theta + D_2\sin\theta\cos^2\theta\right]$$

For $\underset{\sim}{c}_g \,//\, \underset{\sim}{c}_p$, ratio of $x:y$ components must be equal:

i.e. $\dfrac{\cos\theta}{\sin\theta} = \dfrac{2D_1\cos^3\theta + D_2\sin^2\theta\cos\theta}{2D_3\sin^3\theta + D_2\sin\theta\cos^2\theta}$

$\therefore$ <u>either</u> $\cos\theta = 0$ <u>or</u> $\sin\theta = 0$

<u>or</u> $\underbrace{2D_1\cos^2\theta + D_2\sin^2\theta = 2D_3\sin^2\theta + D\cos^2\theta}$ (2)

[20%]

So $\underset{\sim}{c}_p \,//\, \underset{\sim}{c}_g$ if $\theta = 0$ or $\pi/2$, but (2) is clearly not satisfied for every possible D_1, D_2, D_3, θ so in general $\underset{\sim}{c}_g$ is in a different direction to $\underset{\sim}{c}_p$.

For $\underset{\sim}{c}_p \perp \underset{\sim}{c}_g$ need $\underset{\sim}{c}_p \cdot \underset{\sim}{c}_g = 0$

ie need $2D_1\cos^4\theta + 2D_2\sin^2\theta\cos^2\theta + 2D_3\sin^4\theta = 0$

$\underbrace{\text{This is a sum of positive terms provided the } D\text{'s are all positive, so it cannot be zero for any value of } \theta.}$

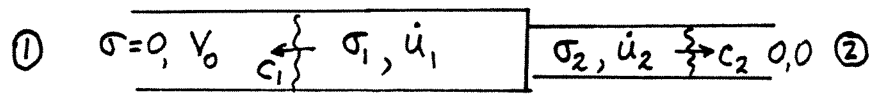
[20%]

② (a) Interface cond.

$$\dot{u}_1(0-,t) = \dot{u}_2(0+,t) \qquad (i)$$

$$\underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim}{A_1\sigma_1(0-,t) = A_2\sigma_2(0+,t)} \qquad (ii) \quad [20\%]$$

(b)



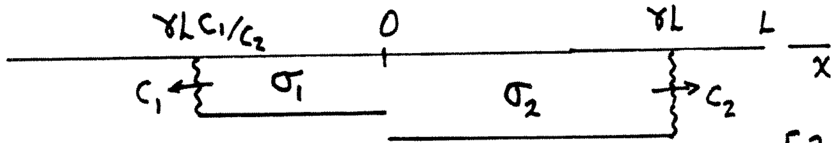
across wavefronts

$$[\sigma_1 - 0] = +\rho_1 c_1[\dot{u}_1 - V_0] \qquad (iii)$$

$$[\sigma_2 - 0] = -\rho_2 c_2[\dot{u}_2 - 0] \qquad (iv)$$

let $\beta = \dfrac{A_2\rho_2 c_2}{A_1\rho_1 c_1}$ and solve (i)-(iv)

$$\dot{u}_1 = \dot{u}_2 = \frac{V_0}{1+\beta}, \quad \sigma_1 = -\frac{\rho_1 c_1 \beta}{1+\beta}V_0, \quad \sigma_2 = -\frac{\rho_2 c_2}{1+\beta}V_0$$



[30%]

(c) initial momentum = momentum at $t = L\gamma/c_2$

$$\rho_1 A_1 L V_0 = \rho_1 A_1\left\{L\left(1-\gamma\frac{c_1}{c_2}\right)V_0 + \gamma\frac{c_1}{c_2}L\frac{V_0}{1+\beta}\right\} + \rho_2 A_2\frac{\gamma L V_0}{1+\beta}$$

$$= \rho_1 A_1 L V_0\left\{1 - \gamma\frac{c_1}{c_2} + \frac{\gamma c_1}{c_2(1+\beta)} + \frac{\gamma c_1\beta}{c_2(1+\beta)}\right\} = \rho_1 A_1 L V_0$$

[20%]

② (d)

Initial Kinetic energy, $\quad 2K = \int \dot{u}^2(x,t)\, \rho A\, dx$

$$2K_o = \rho_1 A_1 L V_o^2$$

Kinetic energy $\quad K(t = \gamma L/c_2)$

$$2K = \rho_1 A_1 L\left(1 - \gamma\frac{c_1}{c_2}\right)V_o^2 + \rho_1 A_1 L \frac{\gamma c_1}{c_2}\left(\frac{V_o}{1+\beta}\right)^2 + \rho_2 A_2 L \gamma\left(\frac{V_o}{1+\beta}\right)^2$$

$$= \rho_1 A_1 L V_o^2 \left\{1 - \gamma\frac{c_1}{c_2} + \gamma\frac{c_1}{c_2}\frac{1}{(1+\beta)^2} + \gamma\frac{c_1}{c_2}\frac{\beta}{(1+\beta)^2}\right\}$$

$$= \rho_1 A_1 L V_o^2 \left\{1 - \frac{\gamma c_1 \beta}{c_2(1+\beta)}\right\}$$

strain energy $\quad U(t = \gamma L/c_2), \quad 2U = \frac{1}{E}\int \sigma(x,t)^2\, A\, dx$

$$2U = \frac{A_1 L}{E_1}\frac{\gamma c_1}{c_2}\left[\frac{\rho_1 c_1 \beta}{1+\beta} V_o\right]^2 + \frac{A_2 L \gamma}{E_2}\left[\frac{\rho_2 c_2}{1+\beta} V_o\right]^2$$

$$= \rho_1 A_1 L V_o^2 \left\{\frac{\gamma c_1}{c_2}\frac{\beta^2}{(1+\beta)^2} + \frac{\gamma c_1}{c_2}\frac{\beta}{(1+\beta)^2}\right\}$$

$$= \rho_1 A_1 L V_o^2 \frac{\gamma c_1}{c_2}\frac{\beta}{(1+\beta)}$$

$\therefore \quad K_o = K + U$ $\qquad$ [30%]

Q3 (a) unidirectional displ. $\Rightarrow$ $u_{x'} = u_{y'} = 0$

Eq. motion: $\rho \dfrac{\partial^2 u_z}{\partial t^2} = \dfrac{\partial \sigma_{z'}}{\partial z'} = 3K \dfrac{\partial^2 u_{z'}}{\partial z'^2}$

$$c_i^2 = \frac{3K_i}{\rho_i} \qquad i = 1, 2$$

[20%]

Displ.

$$u_{z'} = A_1 e^{ik_1(c_1 t - z')} = A_1 e^{ik_1(c_1 t - x\sin\theta_1 - z\cos\theta_1)}$$
$$= A_1 e^{ik_{1x}(ct - x - z\sqrt{\alpha^2 - 1})}$$

where $\alpha = c/c_1 = \dfrac{k_1}{k_{1x}} = \csc\theta_1$, $\alpha^2 - 1 = \cot^2\theta_1$

(b) Snell's law: $\dfrac{\sin\theta_1}{c_1} = \dfrac{\sin\theta_2}{c_2} = \dfrac{\sin\theta_2}{c_1\sqrt{3}} \Rightarrow \theta_2 = \sin^{-1}(\sqrt{3}\sin\theta_1)$

on $z = 0$, $k_{1x} = k_{2x}$ or $k_1 \sin\theta_1 = k_2 \sin\theta_2 = k_2\sqrt{3}\sin\theta_1$

$\therefore$ $k_2 = k_1/\sqrt{3}$, $c_2 = c_1\sqrt{3}$ [20%]

(c) Bdy cond at $z = 0$; noting $\sigma_m = \sigma_{z'} = 3K \dfrac{\partial u_{z'}}{\partial z'}$

$$\begin{cases} \sigma_{mI}(x, 0-, t) + \sigma_{mR}(x, 0-, t) = \sigma_{mT}(x, 0+, t) \\ u_{zI}(x, 0-, t) + u_{zR}(x, 0-, t) = u_{zT}(x, 0+, t) \end{cases}$$

noting $\sigma_m = \sigma_{z'} = 3K\, \partial u_{z'}/\partial z'$, conditions above give

$$\begin{cases} -3Kk_1 (A_1 - A_2) = -3K k_2 C_2 \\ \cos\theta_1 (A_1 + A_2) = (\cos\theta_2) C_2 \end{cases} \rightarrow \begin{cases} A_1 - A_2 = \frac{1}{\sqrt{3}} C_2 \\ A_1 + A_2 = \phi(\theta_1) C_2 \end{cases}$$

$$\phi = \frac{\sqrt{1 - 3\sin^2\theta_1}}{\sqrt{1 - \sin^2\theta_1}}$$

[20%]

(d) solving, $\dfrac{A_2}{A_1} = \dfrac{-1}{1 + \phi\sqrt{3}}$, $\dfrac{C_2}{A_1} = \dfrac{\sqrt{3}}{1 + \phi\sqrt{3}}$ [25%]

(e) Sol'n valid for $\Re\, \phi$ (i.e. no surface waves); $\theta_1 < \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

[15%]