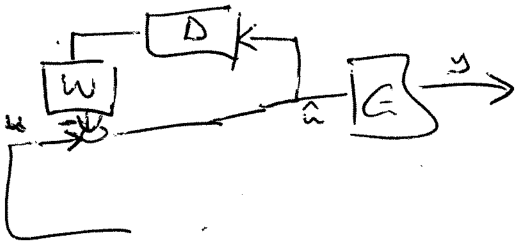
1) a)

is stable for all G satisfying $G \in \mathcal{H}_\infty$, $\|G\|_\infty < 1$ if, and only if, $K \in \mathcal{H}_\infty$ and $\|K\|_\infty \le 1$.

$\mathcal{H}_\infty$ is the space of transfer function matrices satisfying $\bar\sigma(G(s)) < \infty \ \forall\ s: \mathrm{Re}(s) > 0$. For $G \in \mathcal{H}_\infty$, $\|G\|_\infty := \sup_{s: \mathrm{Re}(s)>0} \bar\sigma(G(s))$

$= \sup_\omega \bar\sigma(G(j\omega))$.

b) let $G_1 = G(I + W\Delta)^{-1}$. This may be represented as



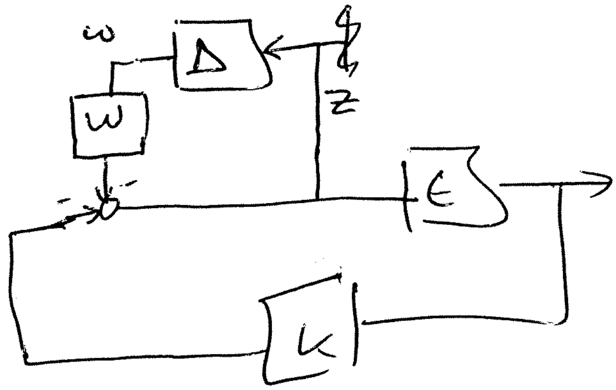
, as $y = G\hat{u}$

$\hat{u} = u - W\Delta\hat{u}$

$\Rightarrow \hat{u} = (I + W\Delta)^{-1} u$

$\Rightarrow y = G(I + W\Delta)^{-1} u$ as required

Adding a controller gives



Now $z = -Ww - KGz$

$\Rightarrow z = -(I + KG)^{-1} W w$

using the small gain theorem, we see that the system is stable for all Δ, $\|\Delta\|_\infty < \epsilon$ iff $\|(I + KG)^{-1} W\|_\infty \le \frac{1}{\epsilon}$.

c) $G_1 = G/(I+W\Delta) \Rightarrow W\Delta = \frac{G}{G_1} - 1$

$$= \frac{\frac{s}{s^2+(0.1\cancel{+\delta_1})s+1\cancel{+\delta_2}}}{\frac{s}{s^2+(0.1+\delta_1)s+1+\delta_2}} - 1$$

$$= \frac{s^2+0.1s+1+\delta_1 s+\delta_2}{s^2+0.1s+1} - 1$$

$$= \frac{\delta_1 s+\delta_2}{s^2+0.1s+1}$$

let $W = \frac{s+1}{s^2+0.1s+1}$

$$\Rightarrow \Delta = \frac{\delta_1 s+\delta_2}{s+1}$$

$$\Rightarrow |\Delta| = \frac{\delta_1^2\omega^2+\delta_2^2}{\omega^2+1} < 1 \text{ iff } |\delta_1|<1 \text{ and } |\delta_2|<1$$

Now $K = 1.9 \Rightarrow \frac{1}{1+KG} = \frac{1}{1+\frac{1.9s}{s^2+0.1s+1}}$

$$= \frac{s^2+0.1s+1}{s^2+2s+1}$$

$$\Rightarrow \frac{W}{1+KG} = \frac{s+1}{s^2+0.1s+1}\cdot\frac{s^2+0.1s+1}{s^2+2s+1} = \frac{1}{s+1}$$

$$\Rightarrow \|\cdot\| = \frac{1}{\sqrt{\omega^2+1}} \le 1$$

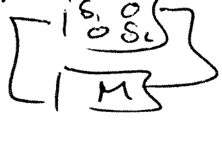
$\Rightarrow$ f/b system is stable using result of (b).

$$\frac{1}{1+KG_1} = \frac{1}{1+\frac{1.9s}{s^2+(0.1+\delta_1)s+1+\delta_2}} = \frac{\sim}{s^2+(2+\delta_1)s+1+\delta_2}$$

$\Rightarrow$ stable for $\delta_1 > \underline{-2}$, $\delta_2 > -1$.

← so small gain is conservative here

Results can be improved using μ, writing loop as



and checking $\mu(M)$

4F2 SOLUTIONS

~~Question 1~~ 2

Question 3(a) From the notes, dropping the s dependence to simplify the notation,

$$\begin{aligned} u &= Ky = K(P_{21}w + P_{22}u) \\ \Rightarrow u &= (I - KP_{22})^{-1}KP_{21}w \\ \Rightarrow z &= P_{11}w + P_{12}u = (P_{11} + P_{12}(I - KP_{22})^{-1}KP_{21})w. \end{aligned}$$

But $K(I - P_{22}K)^{-1} = (I - KP_{22})^{-1}K$, therefore

$$\mathcal{F}_l(P(s), K(s)) = P_{11}(s) + P_{12}(s)K(s)(I - P_{22}(s)K(s))^{-1}P_{21}(s).$$

(b)(i) This is the stadard $\mathcal{H}_2$ optimal control framework with state feedback. The CARE is

$$XA + A^TX + C_1^TC_1 - XB_2B_2^TX = 0.$$

Since X is symmetric let

$$X = \begin{bmatrix} a & b \\ b & d \end{bmatrix}.$$

Substituting the values for A, C_1, B_2 for the system leads to

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ b & d \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} a & b \\ b & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ b & d \end{bmatrix} = 0$$

This leads to the system of equations

$$\begin{aligned} 4 - b^2 &= 0 \\ a - bd &= 0 \\ 2b - d^2 &= 0. \end{aligned}$$

The first equation implies that $b = \pm 2$. Substituting into the third equation shows that only $b = 2$ leads to a real solution, with $d = \pm 2$. Substituting into the second equation leads to $a = \pm 4$. The two real symmetric solutions of the CARE are

$$X_1 = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} -4 & 2 \\ 2 & -2 \end{bmatrix}.$$

(b)(ii) $u = -B_2^TXx$. Substituting gives two controllers

$$\begin{aligned} u_1 &= -\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}x = \begin{bmatrix} -2 & -2 \end{bmatrix}x \\ u_2 &= -\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} -4 & 2 \\ 2 & -2 \end{bmatrix}x = \begin{bmatrix} -2 & 2 \end{bmatrix}x. \end{aligned}$$

(b)(iii) The closed loop state space equations with controller u_1 are

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}x + \begin{bmatrix} 1 \\ 0 \end{bmatrix}w + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} -2 & -2 \end{bmatrix}x = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}x + \begin{bmatrix} 1 \\ 0 \end{bmatrix}w \\ z &= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} -2 & -2 \end{bmatrix}x = \begin{bmatrix} 2 & 0 \\ -2 & -2 \end{bmatrix}x. \end{aligned}$$

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The closed loop system is stable if the roots of

$$det\left(\lambda I - \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\right) = \lambda^2 + 2\lambda + 2$$

have negative real parts. The roots are $\lambda = -1 \pm j$, therefore u_1 is stabilizing.

Likewise the closed loop system for u_2 is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w$$

$$z = \begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix} x.$$

The characteristic polynomial this time is $\lambda^2 - 2\lambda + 2 = 0$ with roots $\lambda = 1 \pm j$. Therefore the closed loop system with controller u_2 is unstable.

(b)(iv)

$$\mathcal{F}_l(P(s), K(s)) = \begin{bmatrix} 2 & 0 \\ -2 & -2 \end{bmatrix} \left(sI - \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\right)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\begin{bmatrix} 2 & 0 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} s+2 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{s^2 + 2s + 2}$$

Therefore

$$\mathcal{F}_l(P(s), K(s)) = \begin{bmatrix} \frac{2(s+2)}{s^2+2s+2} \\ \frac{-2s}{s^2+2s+2} \end{bmatrix}.$$

(b)(v)

$$\min_{K(s) \text{ stabilizing}} \|\mathcal{F}_l(P(s), K(s))\|_2 = \sqrt{2\pi\, trace(B_1^T X_1 B_1)},$$

where

$$B_1^T X_1 B_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 4$$

Therefore

$$\min_{K(s) \text{ stabilizing}} \|\mathcal{F}_l(P(s), K(s))\|_2 = 2\sqrt{2\pi}.$$

Question 2 (a)(i) Notice that $\frac{\partial V}{\partial t}(x,t) = \dot{p}(t)x^2$ and $\frac{\partial V}{\partial x}(x,t) = 2p(t)x$. Substituting into Isaacs equation leads to

$$\dot{p}(t)x^2 + \max_{w\in\mathbb{R}} \min_{u\in\mathbb{R}} \left(u^2 - w^2 + 2p(t)x(x+u+w)\right) = 0.$$

Rearranging, this simplifies to

$$(\dot{p}(t) + 2p(t))x^2 + \max_{w\in\mathbb{R}}(2p(t)xw - w^2) + \min_{u\in\mathbb{R}}(u^2 + 2p(t)xu) = 0.$$

The boundary condition is $V(x,1) = x^2$.

(a)(ii) The minimum with respect to u occurs at $u = -p(t)x$, with $\min_u(u^2 + 2p(t)xu) = -p(t)^2x^2$. The maximum with respect to w occurs at $w = p(t)x$, with $\max_w(-w^2 + 2p(t)xw) = p(t)^2x^2$. Substituting into Isaacs equation gives

$$(\dot{p}(t) + 2p(t))x^2 + p(t)^2x^2 - p(t)^2x^2 = 0.$$

The Riccati equation is therefore

$$\dot{p}(t) + 2p(t) = 0.$$

Notice that the equation is degenerate, in the sense that all quadratic terms have been canceled and what remains is a linear differential equation.

(a)(iii) The solution of the linear differential equation

$$\dot{p}(t) = -2p(t)$$

has the form $p(t) = e^{-2t}p(0)$. The only problem is that we do not know $p(0)$. However

$$V(x,1) = p(1)x^2 = J_1(x) = x^2.$$

Therefore, $p(1) = 1$, which implies that $p(0) = e^2$. Therefore,

$$V(x,t) = e^{-2(t-1)}x^2.$$

(b)(i) Isaacs equation is

$$\frac{\partial V}{\partial t}(x,t) + \max_{w\in[-1,1]} \frac{\partial V}{\partial x}(x,t)w + \min_{u\in[-1,1]} \frac{\partial V}{\partial x}(x,t)(1+|x|)u = 0$$

with boundary condition $V(x,1) = 1 - x^2$.

(b)(ii) Notice that for the proposed solution

$$V(x,1) = 1 - x^2e^{-2(1-1)} = 1 - x^2 = J_1(x).$$

Therefore the proposed solution satisfies the boundary condition. Moreover,

$$\frac{\partial V}{\partial t}(x,t) = 2x^2e^{-2(t-1)} \text{ and } \frac{\partial V}{\partial x}(x,t) = -2xe^{-2(t-1)}$$

Substituting into Isaacs equation we get

$$2x^2e^{-2(t-1)} + \max_{w\in[-1,1]} \left(-2xe^{-2(t-1)}\right) w + \min_{u\in[-1,1]} \left(-2xe^{-2(t-1)}\right)(1+|x|)u = 0.$$

As suggested, distinguish cases $x > 0$, $x = 0$, $x < 0$.

$$\max_{w\in[-1,1]} \left(-2xe^{-2(t-1)}\right) w = \begin{cases} 2xe^{-2(t-1)} & \text{if } x > 0, \text{ for } w = -1 \\ 0 & \text{if } x = 0, \text{ for } w \text{ arbitrary} \\ -2xe^{-2(t-1)} & \text{if } x < 0, \text{ for } w = 1. \end{cases}$$

$$\min_{u\in[-1,1]} \left(-2xe^{-2(t-1)}\right)(1+|x|)u = \begin{cases} -2xe^{-2(t-1)}(1+x) & \text{if } x > 0, \text{ for } u = 1 \\ 0 & \text{if } x = 0, \text{ for } u \text{ arbitrary} \\ 2xe^{-2(t-1)}(1-x) & \text{if } x < 0, \text{ for } u = -1. \end{cases}$$

Isaacs equation is trivially satisfied if $x = 0$. If $x > 0$ it becomes

$$2x^2e^{-2(t-1)} + 2xe^{-2(t-1)} - 2xe^{-2(t-1)}(1+x) = 0.$$

OK. If $x < 0$ it becomes

$$2x^2e^{-2(t-1)} - 2xe^{-2(t-1)} + 2xe^{-2(t-1)}(1-x) = 0.$$

OK again!