4F3 SOLUTIONS

Question 1(a) Consider $e = E\sin(\theta)$. If $E \le 1$, $sat(e) = e$, therefore $N(E) = 1$. If $E > 1$,

$$N(E) = \frac{U_1 + jV_1}{E}.$$

$V_1 = 0$ since $sat(e)$ is an odd function.

$$\begin{aligned}
U_1 &= \frac{1}{\pi}\int_0^{2\pi} sat(E\sin\theta)\sin\theta d\theta = \frac{4}{\pi}\int_0^{\pi/2} sat(E\sin\theta)\sin\theta d\theta \\
&= \frac{4}{\pi}\int_0^{\sin^{-1}(1/E)} E\sin^2\theta d\theta + \frac{4}{\pi}\int_{\sin^{-1}(1/E)}^{\pi/2} \sin\theta d\theta \\
&= \frac{4E}{\pi}\int_0^{\sin^{-1}(1/E)} \frac{1-\cos(2\theta)}{2} d\theta - \frac{4}{\pi}\left[\cos\theta\right]_{\sin^{-1}(1/E)}^{\pi/2} \\
&= \frac{2E}{\pi}\left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{\sin^{-1}(1/E)} + \frac{4}{\pi}\cos\left(\sin^{-1}\left(\frac{1}{E}\right)\right) \\
&= \frac{2E}{\pi}\left[\sin^{-1}\left(\frac{1}{E}\right) - \frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{1}{E}\right)\right)\right] + \frac{4}{\pi}\cos\left(\sin^{-1}\left(\frac{1}{E}\right)\right) \\
&= \frac{2E}{\pi}\left[\sin^{-1}\left(\frac{1}{E}\right) - \sin\left(\sin^{-1}\left(\frac{1}{E}\right)\right)\cos\left(\sin^{-1}\left(\frac{1}{E}\right)\right)\right] + \frac{4}{\pi}\cos\left(\sin^{-1}\left(\frac{1}{E}\right)\right) \\
&= \frac{2E}{\pi}\sin^{-1}\left(\frac{1}{E}\right) + \frac{2}{\pi}\cos\left(\sin^{-1}\left(\frac{1}{E}\right)\right) \\
&= \frac{2E}{\pi}\sin^{-1}\left(\frac{1}{E}\right) + \frac{2}{\pi}\frac{\sqrt{E^2-1}}{E}.
\end{aligned}$$

Therefore,

$$N(E) = \begin{cases} 1 & \text{if } E \le 1 \\ \frac{2}{\pi}\left[\sin^{-1}\left(\frac{1}{E}\right) + \frac{1}{E}\sqrt{1-\frac{1}{E^2}}\right] & \text{if } E > 1. \end{cases}$$

(b) A limit cycle is predicted if

$$-\frac{1}{N(E)} = G(j\omega),$$

where

$$G(s) = \frac{k}{s(s+1)^2}.$$

$N(E)$ is real and $0 < N(E) \le 1$. Therefore,

$$-\infty < -\frac{1}{N(E)} \le -1.$$

So a limit cycle is predicted if $G(j\omega)$ intersects the real axis to the left of -1.

$$G(j\omega) = \frac{k}{j\omega(j\omega+1)^2}$$

has phase

$$-\frac{\pi}{2} - 2\tan^{-1}(\omega)$$

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which is equal to $-\pi$ when

$$2\tan^{-1}(\omega_0) = \frac{\pi}{2} \Rightarrow \tan^{-1}(\omega_0) = \frac{\pi}{4}.$$

Therefore the predicted frequency of oscillation is $\omega_0 = 1\text{rad/sec}$.

To obtain oscillations at this frequency we need $|G(j1)| > 1$.

$$|G(j1)| = \frac{k}{1 \cdot |j+1|^2} = \frac{k}{2}.$$

Therefore and oscillation is obtained if $k > 2$.

(c) The saturation nonlinearity lies in the sector $[0,1]$. Therefore the circle of the circle criterion becomes a vertical line through -1 in the complex plane. Hence, we will have global asymptotic stability if

$$Re[G(j\omega)] > -1 \text{ for all } \omega.$$

Now

$$\begin{aligned} G(j\omega) &= \frac{k}{j\omega(1-\omega^2+2j\omega)} \\ &= \frac{k}{j\omega(1-\omega^2)-2\omega^2} \\ &= -k\frac{j\omega(1-\omega^2)+2\omega^2}{\omega^2(1-\omega^2)^2+4\omega^4}. \end{aligned}$$

Therefore

$$Re[G(j\omega)] = \frac{-2k\omega^2}{\omega^2(1-\omega^2)^2+4\omega^4} = \frac{-2k}{(1-\omega^2)^2+4\omega^2}.$$

So we need

$$\frac{2k}{(1-\omega^2)^2+4\omega^2} < 1 \text{ for all } \omega.$$

If we let $x = \omega^2$ this is equivalent to

$$(1-x)^2 + 4x - 2k > 0$$

for all $x \geq 0$. This is equivalent to

$$x^2 + 2x + 1 - 2k = (x+1)^2 - 2k > 0$$

This quantity is minimized for $x \geq 0$ when $x = 0$. Therefore we need

$$1 - 2k > 0 \Rightarrow k < \frac{1}{2}.$$

Question 2(a) The dimension of the system is $n = 2$.

(b) When $u = 0$ the equations of the system become

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + (1-x_1^2)x_2. \end{aligned}$$

To find the equilibria set the first equation to zero to get $x_2 = 0$, then substitute into the second to get $x_1 = 0$. Therefore, there is only one equilibrium, at $x_1 = x_2 = 0$.

The linearisation about this equilibrium is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} x.$$

The eigenvalues of the matrix are given by

$$\begin{vmatrix} \lambda & -1 \\ 1 & \lambda - 1 \end{vmatrix} = \lambda^2 - \lambda + 1 = 0.$$

Therefore $\lambda = (1 \pm j\sqrt{3})/2$. The eigenvalues have positive real part, hence the equilibrium is unstable.

(c) The closed loop system equations are

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + (1 - x_1^2)x_2 + (1 + x_1^2 + x_2^2)(ax_1 + bx_2).$$

(i) The first equation again gives $x_2 = 0$. The second equation then becomes

$$-x_1 + (1 + x_1^2)ax_1 = (a-1)x_1 + ax_1^3 = x_1((a-1) + ax_1^2) = 0.$$

Therefore, $x_1 = 0$ or $(a-1) + ax_1^2 = 0$. If $0 < a < 1$ there are three equilibria, $(0,0)$, $(\pm\sqrt{(1-a)/a}, 0)$. If $a \leq 0$ or $a \geq 1$ then there is only one equilibrium $(0,0)$.

(ii) For $a = 0$ the system equations simplify to

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + (1 - x_1^2)x_2 + (1 + x_1^2 + x_2^2)bx_2.$$

The linearisation about $(0,0)$ is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 1+b \end{bmatrix} x.$$

The eigenvalues of the matrix are given by

$$\begin{vmatrix} \lambda & -1 \\ 1 & \lambda - (1+b) \end{vmatrix} = \lambda^2 - \lambda(1+b) + 1 = 0.$$

Any value of $b < -1$ will make $(0,0)$ locally asymptotically stable (e.g. choose $b = -3$ to put both poles of the linearisation at $\lambda = -1$).

(d) The controller your friend proposes will lead to the closed loop system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + (1 - x_1^2)x_2 + (1 + x_1^2 + x_2^2)\frac{x_1^2 x_2 - 3x_2}{1 + x_1^2 + x_2^2} = -x_1 + x_2 - x_1^2 x_2 + x_1^2 x_2 - 3x_2 = -x_1 - 2x_2.$$

Notice that the closed loop system is linear, therefore is $(0,0)$ is globally asymptotically stable if and only if the eigenvalues of the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

have negative real parts. The calculation is the same as above, there are two eigenvalues at $\lambda = -1$. Your friend is right!

Question 3(a)(i) A state $(\hat{q}, \hat{x})$ is called reachable if the automaton can find itself at that state along one of its executions. More formally, $(\hat{q}, \hat{x})$ is reachable if and only if there exists an initial state $(q_0, x_0) \in$ *Init* and an execution (τ, q, x) starting at (q_0, x_0) (more formally, $(q_0(\tau_0), x_0(\tau_0)) = (q_0, x_0)$) such that $(\hat{q}, \hat{x}) = (q_N(\tau_N'), x_N(\tau_N'))$.

(ii) A set of states M is called invariant if all executions that start in this set remain in this set for ever. More formally, M is invariant if and only if for all $(q_0, x_0) \in M$, for all (τ, q, x) starting at (q_0, x_0), for all $[\tau_i, \tau_i'] \in \tau$ and for all $t \in [\tau_i, \tau_i']$, $(q_i(t), x_i(t)) \in M$. Notice that it is not necessary for (q_0, x_0) to be in *Init*.

(iii) Since *Init* $\subseteq M$, all executions of the system start in M. Since M is invariant, the state remains in M for ever. Therefore, all reachable states are in M. Since $M \subseteq F$, all reachable states are also in F.

(b) (i) $Q = \{\textit{Left}, \textit{Right}\}$, $X = \mathbb{R}^2$,

$$\textit{Init} = \{\textit{Left}\} \times \left\{ x \in X \;\middle|\; \left(\cos(1 + x_1) + \frac{x_2^2}{2} \leq 1\right) \text{ and } (x_1 \leq 0) \right\} \cup$$
$$\{\textit{Right}\} \times \left\{ x \in X \;\middle|\; \left(\cos(1 - x_1) + \frac{x_2^2}{2} \leq 1\right) \text{ and } (x_1 \leq 0) \right\}.$$

(ii) Since $(q_0, x_0) \in \mathit{Init}$ and $q_0 = \mathit{Left}$ then $x_0 = x(0)$ is such that $\cos(1 + x_1(0)) + \frac{x_2(0)^2}{2} \leq 1$ and $x_1(0) \leq 0$. The only way to leave *Init* along the solution of the differential equation is if at some time $t \geq 0$ either $x_1(t) > 0$, or $\cos(1 + x_1(t)) + \frac{x_2(t)^2}{2} > 1$. The former is impossible since $x(t) \in \mathit{Dom}(q_0) = \{x \in X \mid x_1 \leq 0\}$. For the latter, notice that

$$\begin{aligned}\frac{d}{dt}\left(\cos(1 + x_1(t)) + \frac{x_2(t)^2}{2}\right) &= -\sin(1 + x_1(t))\dot{x}_1(t) + x_2(t)\dot{x}_2(t) \\ &= -\sin(1 + x_1(t))x_2(t) + x_2(t)\sin(1 + x_1(t)) \\ &= 0.\end{aligned}$$

Therefore, along solutions of the differential equation $\cos(1 + x_1(t)) + \frac{x_2(t)^2}{2} = \cos(1 + x_1(0)) + \frac{x_2(0)^2}{2}$ and $(q_0, x(t)) \in \mathit{Init}$ as long as $x(t) \in \mathit{Dom}(q_0)$.

(iii) Since $(q_0, x_0) \in \mathit{Init}$ and $q_0 = \mathit{Left}$ then x_0 must be such that $x_1 \leq 0$. If a discrete transition $\mathit{Left} \to \mathit{Right}$ takes place from (q_0, x_0) then

$$x_0 \in G(\mathit{Left}, \mathit{Right}) = \{x \in X \mid x_1 \geq 0\}.$$

Therefore, $x_1 = 0$. Since $(q_0, x_0) \in \mathit{Init}$ and $q_0 = \mathit{Left}$ then x_0 must also be such that $\cos(1 + x_1) + \frac{x_2^2}{2} = \cos(1) + \frac{x_2^2}{2} \leq 1$.

The discrete transition does not affect x_1 and makes x_2 half of what it was. Therefore, after the discrete transition $q = \mathit{Right}$, $x_1 = 0$ and $\cos(1) + \frac{x_2^2}{2} \leq 1$ (since this was true before the transition and x_2 is now smaller). Therefore the state is still in *Init*.

(iv) Parts (b)(ii) and (b)(iii) imply that the state can not leave the set *Init* either along continuous evolution or along a discrete transition. Therefore the set *Init* is invariant. Clearly $\mathit{Init} \subseteq \mathit{Init}$. Therefore, by (a)(iii), the set of reachable states is contained in *Init*. But, by definition, all states in *Init* are reachable, therefore, the set of reachable states is *Init* itself.

Question 4(a) Model checking is a method for automatically verifying properties of a hybrid system using a computer. The model of the system and the specification are given as input to a program which analyses the system and tells the user whether it meets the specification or not. If the system does not meet the specification, a solution of the system that fails the specification (known as a *witness*) is produced.

Timed automata are a special class of hybrid automata for which:

- The continuous dynamics are of the form

$$\dot{x} = f(q, x) = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

  (*clocks*).
- All guards (G), domains (Dom) and the set of initial conditions (Init) are given by rectangular sets.
- The reset relation (R) is either given by a rectangular set, or may leave some of the continuous variables unchanged.

Timed automata are amenable to model checking methods because they admit finite bisimulations.

(b)(i) The dimension of the system is $n = 1$.

(ii) The equilibria of the system are the values of x solving the equation

$$\mu x - x^3 = x(\mu - x^2) = 0.$$

If $\mu > 0$ this equation has thee solutions, $x = 0$ and $x = \pm\sqrt{\mu}$. If $\mu \leq 0$ the equation has only one solution, $x = 0$.

(iii) The linearisation of the system about and equilibrium $\hat{x}$ is

$$\dot{x} = (\mu - 3\hat{x}^2)x.$$

If $\mu > 0$ there are three equilibria, $\hat{x} = 0$ and $\hat{x} = \pm\sqrt{\mu}$. The linearisation about $\hat{x} = 0$ is

$$\dot{x} = \mu x.$$

Since $\mu > 0$ this equilibrium is unstable. The linearisation about the equilibria $\hat{x} = \pm\sqrt{\mu}$ is

$$\dot{x} = -2\mu x.$$

Since $\mu > 0$ these equilibria are stable.

If $\mu < 0$ there is only one equilibrium at $\hat{x} = 0$. The linearisation about this equilibrium is again

$$\dot{x} = \mu x.$$

Since $\mu < 0$ this equilibrium is stable.

(iv) The plot is shown in Figure 1. The dotted lines represent unstable equilibria, the solid lines stable ones.

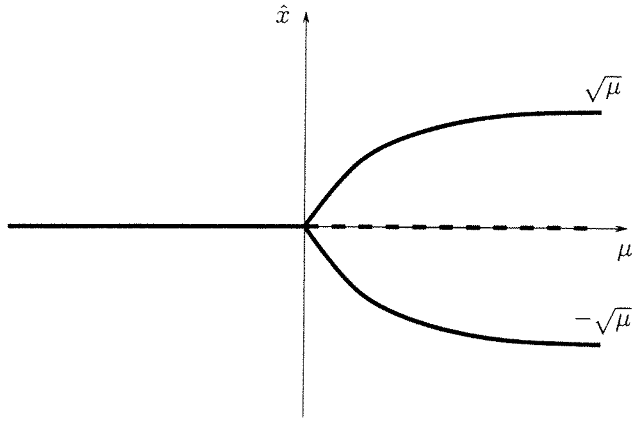

FIGURE 1. Equilibria for Question 3(b)(iv)