Q1

1. Replacing X_i in $Z_i = X_i + N_i$ we have

$$Z_i = \alpha^i \Theta + N_i \longrightarrow Z_i|\Theta = \theta \sim \mathcal{N}\left(\alpha^i\theta, \sigma^2\right)$$

$$f(\underline{Z}|\theta) = \prod_{i=1}^{n} (2\pi\sigma^2)^{-1/2} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(Z_i - \alpha^i\theta)^2\right]$$

$$\begin{aligned}\pi(\theta|\underline{Z}) &\propto \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(Z_i - \alpha^i\theta)^2 - \frac{1}{2q^2}\theta^2\right] \\ &\propto \exp\left[-\frac{1}{2\sigma^2}\theta^2\left[\sum_{i=1}^{n}\alpha^{2i} + \frac{\sigma^2}{q^2}\right] - \frac{1}{2\sigma^2}\sum_{i=1}^{n} Z_i\alpha^i\theta\right] \\ &\propto \exp\left[-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}\alpha^{2i} + \frac{\sigma^2}{q^2}\right]\left(\theta - \widehat{\theta}_{MAP}\right)^2\right] \\ &\propto \exp\left[-\frac{1}{2\,e_n}\left(\theta - \widehat{\theta}_{MAP}\right)^2\right]\end{aligned}$$

$$\widehat{\theta}_{MAP} = \widehat{\theta}_{MMSE} = \frac{q^2\sum_{i=1}^{n}\alpha^i Z_i}{\sigma^2 + q^2\sum_{i=1}^{n}\alpha^{2i}}$$

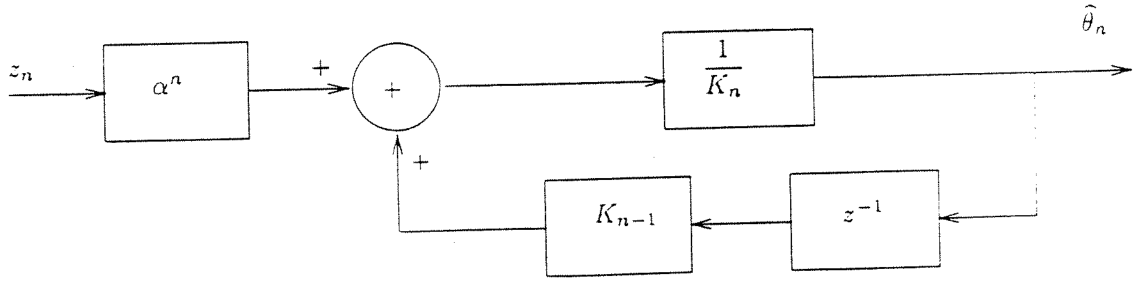
2. Noting

$$\widehat{\theta}_n = \frac{q^2\sum_{i=1}^{n}\alpha^i Z_i}{\sigma^2 + q^2\sum_{i=1}^{n}\alpha^{2i}} = \frac{\sum_{i=1}^{n}\alpha^i Z_i}{\frac{\sigma^2}{q^2} + \sum_{i=1}^{n}\alpha^{2i}}$$

it is easily seen that $\widehat{\theta}_0 = 0$, $K_n = \frac{\sigma^2}{q^2} + \sum_{i=1}^{n}\alpha^{2i}$, $K_0 = \frac{\sigma^2}{q^2}$, $K_n = K_{n-1} + \alpha^{2n}$ and

$$\widehat{\theta}_n = K_n^{-1}\left[K_{n-1}\widehat{\theta}_{n-1} + \alpha^n Z_n\right]$$

3. Block diagram



4. MSE is the variance of the posterior law $\pi(\theta|\underline{Z})$

$$e_n = \frac{\sigma^2}{\sum_{i=1}^n \alpha^{2i} + \frac{\sigma^2}{q^2}} = \sigma^2 K_n^{-1}$$

$$\begin{aligned} K_n &= \frac{\sigma^2}{q^2} + \sum_{i=1}^n \alpha^{2i} \\ e_n &= q^2\sigma^2 \left(\sigma^2 + q^2 \sum_{i=1}^n \alpha^{2i}\right)^{-1} \\ \widehat{\theta}_n &= \frac{q^2 \sum_{i=1}^n \alpha^i Z_i}{\sigma^2 + q^2 \sum_{i=1}^n \alpha^{2i}} \end{aligned}$$

5. Limit cases :

$$q^2 \mapsto \infty \longrightarrow \begin{cases} K_n^{-1} & \mapsto & 0 \\ e_n & \mapsto & 0 \\ \widehat{\theta}_n & \mapsto & 0 \end{cases} , \quad \sigma^2 \mapsto 0 \longrightarrow \begin{cases} K_n^{-1} & \mapsto & \left(\sum_{i=1}^n \alpha^{2i}\right)^{-1} \\ e_n & \mapsto & 0 \\ \widehat{\theta}_n & \mapsto & \frac{\sum_{i=1}^n \alpha^i Z_i}{q^2 \sum_{i=1}^n \alpha^{2i}} \end{cases}$$

$$K_\infty = \begin{cases} \sigma^2 + \frac{q^2\alpha^2}{1-\alpha^2} & \text{if } \alpha < 1 \\ \infty & \text{if } \alpha \geq 1 \end{cases}$$

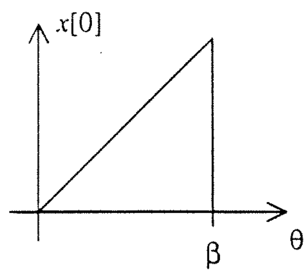
$$e_\infty = \begin{cases} \frac{q^2\sigma^2}{\sigma^2 + \frac{q^2\alpha^2}{1-\alpha^2}} & \text{if } \alpha < 1 \\ 0 & \text{if } \alpha \geq 1 \end{cases}$$

$$\widehat{\theta}_\infty = \begin{cases} \frac{q^2 \sum_{i=1}^n \alpha^i Z_i}{\sigma^2 + \frac{q^2\alpha^2}{1-\alpha^2}} & \text{if } \alpha < 1 \\ 0 & \text{if } \alpha \geq 1 \end{cases}$$

2. **Part a:**

$$p\left(x[0],\theta\right) = p\left(x[0]|\theta\right)p\left(\theta\right) = \begin{cases} \frac{1}{\beta}\frac{1}{\theta}, & 0 \leq x[0] \leq \theta \leq \beta \\ 0, & \text{otherwise} \end{cases}$$

The region where this joint density is non-zero is the following:



Integrating over θ gives the marginal density for $x[0]$:

$$\begin{aligned} p\left(x[0]\right) &= \int p\left(x[0],\theta\right)d\theta \\ &= \begin{cases} \int_{x[0]}^{\beta}\frac{1}{\beta}\frac{1}{\theta}d\theta, & 0 \leq x[0] \leq \beta \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{\beta}\left[\ln(\beta) - \ln(x[0])\right], & 0 \leq x[0] \leq \beta \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

$$\begin{aligned} p\left(\theta|x[0]\right) &= \frac{p\left(x[0],\theta\right)}{p\left(x[0]\right)} \\ &= \begin{cases} \frac{\frac{1}{\beta}\frac{1}{\theta}}{\frac{1}{\beta}\left[\ln(\beta)-\ln(x[0])\right]}, & 0 \leq x[0] \leq \theta \leq \beta \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

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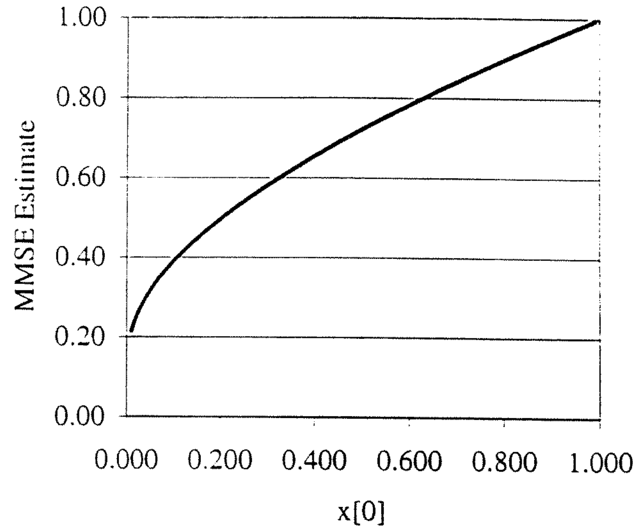
$$= \begin{cases} \frac{\frac{1}{\theta}}{\ln(\beta)-\ln(x[0])}, & 0 \le x[0] \le \theta \le \beta \\ 0, & \text{otherwise} \end{cases}$$

Part b: The MMSE estimate of θ is

$$\hat{\theta}_{\text{MMSE}} = E\left[\theta | x[0]\right] = \int_x^\beta \theta \frac{\frac{1}{\theta}}{\ln(\beta) - \ln(x[0])} d\theta = \frac{\beta - x[0]}{\ln(\beta) - \ln(x[0])}$$

A plot of this estimate for $\beta = 1$ is the following:



The MAP estimate of θ is

$$\hat{\theta}_{\text{MAP}} = \arg\max_\theta p\left(\theta | x[0]\right) = x[0]$$

Part c: For $N > 1$, we get the following:

$$\begin{aligned} p\left(x[0], \ldots, x[N-1], \theta\right) &= \left[\prod_{n=0}^{N-1} p\left(x[n]|\theta\right)\right] p\left(\theta\right) \\ &= \begin{cases} \frac{1}{\beta}\frac{1}{\theta^N}, & 0 \le \max_n x[n] \le \theta \le \beta \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

$$p\left(x[0], \ldots, x[N-1]\right) = \int p\left(x[0], \ldots, x[N-1], \theta\right) d\theta$$

Part c:

$$
\begin{aligned}
p(x[0], \ldots, x[N-1]; \theta) &= \prod_{n=0}^{N-1} p(x[n]; \theta) \\
&= \prod_{n=0}^{N-1} \left[\frac{x[n]}{\theta} e^{-\frac{x^2[n]}{2\theta}} \right] \\
&= \frac{1}{\theta^N} \left[\prod_{n=0}^{N-1} x[n] \right] \exp\left(-\frac{1}{2\theta} \sum_{n=0}^{N-1} x^2[n] \right)
\end{aligned}
$$

$$
\ln p(x[0], \ldots, x[N-1]; \theta) = -N \ln\theta + \left[\sum_{n=0}^{N-1} \ln x[n] \right] - \frac{1}{2\theta} \sum_{n=0}^{N-1} x^2[n]
$$

Taking the derivative with respect to θ and setting it equal to zero gives

$$
\frac{\partial \ln p(x[0], \ldots, x[N-1]; \theta)}{d\theta} = -\frac{N}{\theta} + \frac{1}{2\theta^2} \sum_{n=0}^{N-1} x^2[n] = 0
$$

Adding the hat and solving for $\hat{\theta}$ gives

$$
\hat{\theta} = \frac{1}{2N} \sum_{n=0}^{N-1} x^2[n]
$$

3. **Part a:** The log-likelihood function is

$$\ln p(x[0];\theta) = \ln x[0] - \ln \theta - \frac{x^2[0]}{2\theta}$$

Taking the derivative with respect to θ and setting it to zero,

$$\frac{\partial p(x[0];\theta)}{\partial \theta} = -\frac{1}{\theta} + \frac{x^2[0]}{2\theta^2} = 0$$

Solving for θ (and adding the hat) gives

$$\frac{x^2[0]}{2\hat{\theta}} = 1$$

$$\hat{\theta} = \frac{1}{2}x^2[0]$$

Part b:

$$E\left[\hat{\theta}\right] = E\left[\frac{1}{2}x^2[0]\right] = \int_0^\infty \frac{1}{2}x^2 \, \frac{x}{\theta} e^{-\frac{x^2}{2\theta}} \, dx = \theta$$

Thus, $\hat{\theta}$ is unbiased.

Part c:

$$\begin{aligned} p(x[0], \ldots, x[N-1]; \theta) &= \prod_{n=0}^{N-1} p(x[n]; \theta) \\ &= \prod_{n=0}^{N-1} \left[\frac{x[n]}{\theta} e^{-\frac{x^2[n]}{2\theta}} \right] \\ &= \frac{1}{\theta^N} \left[\prod_{n=0}^{N-1} x[n] \right] \exp\left(-\frac{1}{2\theta} \sum_{n=0}^{N-1} x^2[n] \right) \end{aligned}$$

$$\ln p(x[0], \ldots, x[N-1]; \theta) = -N \ln \theta + \left[\sum_{n=0}^{N-1} \ln x[n] \right] - \frac{1}{2\theta} \sum_{n=0}^{N-1} x^2[n]$$

Taking the derivative with respect to θ and setting it equal to zero gives

$$\frac{\partial \ln p(x[0], \ldots, x[N-1]; \theta)}{d\theta} = -\frac{N}{\theta} + \frac{1}{2\theta^2} \sum_{n=0}^{N-1} x^2[n] = 0$$

Adding the hat and solving for $\hat{\theta}$ gives

$$\hat{\theta} = \frac{1}{2N} \sum_{n=0}^{N-1} x^2[n]$$

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First part is book work.

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The general linear model may be written

$$\underline{d} = \underline{G}\,\underline{\theta} + \underline{w}$$

where the model parameters are contained in the vector $\underline{\theta}$. $\underline{d}$ is the observed data vector, $\underline{w}$ is the noise vector and $\underline{G}$ is a matrix.

~~For~~ the signal

$$s(n) = A + Bn$$

the observed data is $\underline{d} = \underline{s} + \underline{w}$

$$\underline{d} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & N-1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} + \underline{w} = \underline{G}\,\underline{\theta} + \underline{w}.$$

4 cont

The likelihoods for the observed noisy data $d(n)$ are

$$p(d|H_0) = \frac{1}{(2\pi)^{N/2}|C|^{1/2}} e^{-\frac{1}{2}d^T C^{-1} d}.$$

$$p(d|H_1) = \frac{1}{(2\pi)^{N/2}|C|^{1/2}} e^{-\frac{1}{2}(d-s)^T C^{-1}(d-s)}.$$

and the NP detector is

$$L(d) = \frac{p(d|H_1)}{p(d|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

$$\therefore L(d) = e^{-\frac{1}{2}\left[-2d^T C^{-1} s + s^T C^{-1} s\right]}$$

$$\therefore d^T C^{-1} s \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2} s^T C^{-1} s + \log(\lambda).$$

$$\therefore \frac{1}{\sigma^2} d^T \underline{H}\, \underline{\theta} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda' \qquad \therefore \frac{1}{\sigma^2} \sum d(n)(A + Bn) \underset{H_0}{\overset{H_1}{\gtrless}} \lambda'$$