ENGINEERING TRIPOS PART IIB

ELECTRICAL AND INFORMATION SCIENCES TRIPOS PART II

CRIBS 2003

Module 4F7

DIGITAL FILTERS AND SPECTRUM ESTIMATION

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

4F7 2003

Worked solutions

ANSWERS

1. (a) The gain of the analogue filter is, by inspection, 0 dB at high frequency ($\omega \to \infty$) and -3dB at ($\omega = \omega_c$).

Apply frequency warping to given digital critical frequencies:

$\omega_C = \tan(0.375\pi) = 2.4142$

$\omega_S = \tan(0.125\pi) = 0.4142$

Hence to give required stopband attenuation we require

$$\frac{1}{1+(\omega_C/\omega_S)^{2N}} < 10^{-3} \Rightarrow \qquad 2N>3.9182$$

hence N=2.

Convert lowpass transfer function to highpass with correct cutoff frequency:

$$H\left(\frac{2.4142}{s}\right) = \frac{s^2}{s^2+2.4142\sqrt{2}s+2.4142^2}$$

and apply Bilinear Transform

$$s = \frac{1-z^{-1}}{1+z^{-1}}$$

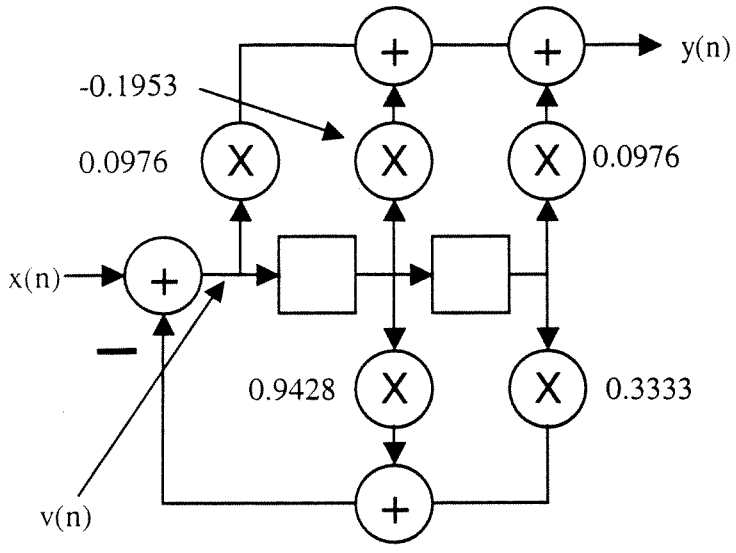
giving

$$G(z) = \frac{\left(1-z^{-1}\right)^2}{\left(1-z^{-1}\right)^2+2.4142\sqrt{2}\left(1-z^{-1}\right)\left(1+z^{-1}\right)+2.4142^2\left(1+z^{-1}\right)^2}$$

$$= \frac{\left(1-z^{-1}\right)^2}{10.2426+9.6569z^{-1}+3.4142z^{-2}}$$

$$= \frac{0.0976\left(1-2z^{-1}+z^{-2}\right)}{1+0.9428z^{-1}+0.3333z^{-2}}$$

(The scaling to make the zeroth-order coefficient of the denominator 1.0 is essential for the implementation, as below, though not essential when simply giving the transfer function)



(c) Overflow occurs when the result of a computation (addition or multiplication) falls outside the number range of the fixed-point arithmetic. The result is an erroneous value - typically a very large error.

Overflow oscillation is an oscillation in a recursive digital filter caused by, and then sustained by, the occurrence of an overflow.

To avoid v(n) becoming larger in magnitude than x(n), using the given fact that the maximum magnitude of the frequency response from x(n) to v(n) is 2.6, the input must be scaled by 1/2.6 prior to the input adder. It would be more normal to scale by a power of two, i.e. 1/4.

From part (a) and (b) the maximum magnitude of the frequency response from x(n) to y(n), ignoring the above scaling, is 1.0. In order to make the output as large as it safely can be (to avoid reducing SNR) the scaling applied at the input should therefore be cancelled, by multiplying the three feedforward terms by 2.6 (or 4, if 1/4 was used at the input).

Frequency response scaling does not absolutely guarantee to prevent overflow, so to prevent overflow oscillation, the arithmetic unit must also implement *saturation arithmetic.*

2 (a) Window method as in lecture notes:

- Select a suitable window function.
- Specify an 'ideal' response $D(\Omega)$ - e.g. ideal ("brickwall") lowpass response
- Compute the coefficients of the "ideal" filter, using the FFT and ensuring the FFT length is sufficient to avoid aliasing.
- Multiply the ideal coefficients by the window function to give the filter coefficients.
- Evaluate the frequency response of the resulting filter, and iterate if necessary.

(b) The design problem.

Use tabulated data to estimate filter length:

transition width/sample frequency = $(0.43\pi - 0.3\pi)/2\pi = 0.065$

Hence estimate N = 2.7/0.065 = 41.5

Method suggested in lecture notes is to start with an ideal "brickwall" filter cutoff frequency in the middle of the transition band, i.e.

$\Omega_c = 0.5(0.43\pi - 0.3\pi) = 0.365\pi = 1.1467$

Taking N=41,and using the formulae in the question ($0.365\pi = 1.1467$, 1.2/(N -10) = -0.0387, 11.5/(N +1) = 0.2738), this gives $\Omega_{-3dB} = 1.1080$ and $\Omega_{-45dB} = 1.4205$
These have an average of 1.2643 and a separation of 0.3125.

But the wanted band edges are $0.3\pi = 0.9425$ and $0.43\pi = 1.3509$, which have an average of 1.1467 and a separation of 0.4084.

The ratio of the separations is 0.4084/0.3125 = 1.3, and since the transition bandwidth is approximately inversely proportional to the filter length N (as mentioned in the lectures and also indicated by the transition width formula in the question), this suggests the filter length could be reduced to 41/1.3 = 31.4.

Perhaps try 31.

It also seems that the centre frequency should be reduced by 1.2643-1.1467 = 0.118.
So try $\Omega_c = 1.1467 - 0.118 = 1.0287$, and N = 31.

This gives $\Omega_{-3dB} = 0.9716$ and $\Omega_{-45dB} = 1.3881$ (compared to 0.9425 and 1.3509)
Actual mean 1.1799 (0.0332 too high) and separation 0.4165 (0.081 too great).

So try $\Omega_c = 1.0287 - 0.0332 = 0.9955$, and N = 33.
This gives $\Omega_{-3dB} = 0.9433$ and $\Omega_{-45dB} = 1.3337$

(meeting the requirements >0.9425 and <1.3509).

Hence the required filter length is 33.

(c) Limits are $0 < \mu < 2/(L\sigma_x^2)$ for convergence to occur.
A suitable value μ_0 to give fast convergence is $1/(L\sigma_x^2)$.

μ (i) slightly higher than μ_0: convergence becomes oscillatory, and slows a little. tracking speed increases, misadjustment noise increases;
(ii) slightly lower, convergence slows a little (not oscillatory),
tracking speed reduces, misadjustment noise reduces;
(iii) much higher, unstable;
(iv) much lower, convergence slows a lot, tracking speed reduces a lot, misadjustment noise reduces more;

If the input signal $x(k)$, still of variance σ_x^2, is coloured, the near-optimum step size has the same value as before, but this corresponds to fast convergence in the direction of the covariance-matrix-eigenvector with largest eigenvalue. Convergence in the direction of the eigenvector with smallest eigenvalue is then slow.

Hence overall convergence rate is slowed.

Adaptive filter algorithms for coloured input signal:

- Not NLMS (not significantly better with coloured input).
- Gradient Adaptive Lattice (GAL) somewhat better. Still order N.
- Recursive Least Squares (RLS) has much faster convergence, and is unaffected by coloured input. However, order N^2 so typically more computation.
- Fast RLS - works as well as RLS but order N, so less computation, though still significantly more than LMS. Unfortunately, all versions are inherently unstable, even with floating-point arithmetic.

3 (a) Periodogram. This is bookwork from the lecture notes, as follows.

Calculate Sample autocorrelation function using the(biased estimate):

$$\boxed{\hat{R}_{XX}[k] = \frac{1}{N}\sum_{n=0}^{N-1-k} x_n x_{n+k} \qquad 0 \le k < N} \tag{1}$$

Note that the autocorrelation is an even function so that estimates for negative k are given by:

$$\hat{R}_{XX}[-k] = \hat{R}_{XX}[k]$$

Now, If the maximum correlation lag is taken to be:

$$L = N - 1$$

then the resulting spectrum estimate is:

$$\hat{S}_X(e^{j\omega T}) = \sum_{k=-(N-1)}^{N-1} \hat{R}_{XX}[k]\, e^{-jk\omega T} \tag{2}$$

When the biased form is used for $\hat{R}_{XX}$, this can be rewritten in terms of the DTFT of $\{x_0, x_1, ..., x_{N-1}\}$:

$$\boxed{\begin{aligned} \hat{S}_X(e^{j\omega T}) &= \frac{1}{N}|X(e^{j\omega T})|^2 \\ X(e^{j\omega T}) &= \sum_{n=0}^{N-1} x_n\, e^{-jn\omega T} \end{aligned}} \tag{3}$$

which is known as the **Periodogram.**

Periodogram is simple to compute with FFTs. The method is asymptotically unbiased. Unfortunately, however, the variance can be very high (and does not necessarily decrease with data length). Hence there are serious fluctuations due to randomness in the signal.

Hence smoothing and averaging techniques are employed - the Bartlett, Blackman-Tukey and Walsh methods.

The Bartlett Procedure

- It would seem natural to try and improve the spectrum estimate by performing some averaging in order to mimic the required ensemble average.

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•Let the data sequence x_n be of length $N_s = KN$ and segment this sequence into K subsequences of length N:

$$x_n^{(k)} = x_{n+kN} \qquad 0 \le n \le N-1 \qquad 0 \le k \le K-1$$

•Calculate the periodogram for each frame, denoted by $\hat{S}_X^{(k)}(e^{j\omega T})$, $k = 1, 2, ..., K$.

•The Bartlett estimate is then given by:

$$\boxed{\hat{S}_X^B(e^{j\omega T}) = \frac{1}{K}\sum_{k=0}^{K-1} \hat{S}_X^{(k)}(e^{j\omega T})} \tag{4}$$

The Blackman-Tukey Procedure

•The Blackman-Tukey method applies a window function of length $2L+1$ to the estimated autocorrelation function:

$$\boxed{\hat{S}_X^{BT}(e^{j\omega T}) = \sum_{-L}^{L} w_l \hat{R}_{XX}[l]\exp(-j\omega T)} \tag{5}$$

where $L < N$ and w_l is any suitable window function, e.g. Hamming, Hanning, Bartlett,...

•We have already analysed a similar case, see page 67. It is clear that the resulting spectrum can be written as a frequency domain convolution:

$$\hat{S}_X^{BT}(e^{j\omega T}) = \frac{1}{2\pi} W(e^{j\omega t}) * \hat{S}_X(e^{j\omega t})$$

where $W(.)$ is the DTFT of the window function and $\hat{S}_X(.)$ is the Periodogram.

•The B-T method can reduce the variance of the periodogram estimate at the expense of some frequency resolution. A special case is the correlogram method.

The Welch Procedure

•The Welch procedure performs averaging over frames as in the Bartlett method

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•However, the periodograms are *modified* to incorporate a window function on the data:

$$\boxed{\hat{S}^{(k)}(e^{j\omega T}) = \frac{1}{N}\left|\sum_{n=0}^{N-1} w_n x_n^{(k)} e^{-j\omega nT}\right|^2}$$

with $1/N\sum_{n=0}^{N-1} w_n^2 = 1$.

•As for the Bartlett method, averaging is then performed over K frames:

$$\boxed{\hat{S}_X^W(e^{j\omega T}) = \frac{1}{K}\sum_{k=0}^{K-1} \hat{S}_X^{(k)}(e^{j\omega T})} \tag{6}$$

•The expected value of this spectral estimate can be shown to be:

$$E[\hat{S}_X^W(e^{j\omega T})] = \frac{1}{2\pi} V(e^{j\omega T}) * S_X(e^{j\omega T})$$

where $W(e^{j\omega T})$ is the DTFT of the window and $V(e^{j\omega T}) = \frac{1}{N}|W(e^{j\omega T})|^2$.

•When the segments are non-overlapping the variance is approximately that of the Bartlett estimate.

[These answers are more detailed than required]

(b)

$$E[\hat{R}_{XX}[k]] = E\left[\frac{1}{N}\sum_{n=0}^{N-1-k} w_n x_n w_{n+k} x_{n+k}\right]$$

$$= \frac{1}{N}\sum_{n=0}^{N-1-k} w_n w_{n+k} E[x_n x_{n+k}]$$

$$= \frac{1}{N}\sum_{n=0}^{N-1-k} w_n w_{n+k} R_{XX}[k]$$

$$= \frac{1}{N} R_{XX}[k] \sum_{n=0}^{N-1-k} w_n w_{n+k}$$

i.e. not unbiased unless $\frac{1}{N}\sum_{n=0}^{N-1-k} w_n w_{n+k} = 1$, which is not true in general.

Now, consider the expected value of the power spectrum estimate:

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•Substituting into the expression for $E[\hat{S}_X(e^{j\omega T})]$ we obtain:

$$\begin{aligned} E[\hat{S}_X(e^{j\omega T})] &= \sum_{k=-(N-1)}^{N-1} E[\hat{R}_{XX}[k]]\, e^{-jk\omega T} \\ &= \sum_{k=-(N-1)}^{N-1} \sum_{n=0}^{N-1-k} w_n w_{n+k} R_{XX}[k]\, e^{-jk\omega T} \\ &= \sum_{k=-(\infty)}^{\infty} \sum_{n=0}^{N-1-k} w_n w_{n+k} R_{XX}[k]\, e^{-jk\omega T} \end{aligned}$$

•This is the DTFT of a *product* of two functions $\sum_{n=0}^{N-1-k} w_n w_{n+k}$ and R_{XX}, equal to the *convolution* of their individual DTFTs:

$$\boxed{E[\hat{S}_X(e^{j\omega T})] = \frac{1}{2\pi}\int_{-\pi}^{\pi} V(e^{j\omega T}) S_X(e^{j(\nu-\omega)T})\, d\nu T} = \frac{1}{2\pi} V(e^{j\omega T}) * S_X(e^{j\omega T}) \tag{7}$$

where $S_X(.)$ is the true power spectrum and $V(.)$ is the DTFT of $\sum_{n=0}^{N-1-k} w_n w_{n+k}$.

Now,

$$\begin{aligned} V(e^{j\omega T}) &= \sum_{k=-\infty}^{+\infty} \sum_{n=0}^{N-1-k} w_n w_{n+k} \exp(-jk\omega T) \\ &= \sum_{k=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} w_n w_{n+k} \exp(-jk\omega T) \text{ [since } w_n = 0 \text{ for } n < 0 \text{ or } n > N-1\text{]} \\ &= \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} w_n w_m \exp(-j(m-n)\omega T) \text{ [set } m = n+k\text{]} \\ &= \sum_{n=-\infty}^{+\infty} w_n \exp(jn\omega T) \sum_{m=-\infty}^{+\infty} w_m \exp(-jm\omega T) \\ &= W^*(e^{j\omega T}) W(e^{j\omega T}) \text{ [since } w_n \text{ assumed real]} \\ &= |W(e^{j\omega T})|^2 \end{aligned}$$

Hence, as required:

$$E[\hat{S}_X(e^{j\omega T})] = \frac{1}{2\pi N} S_X(e^{j\omega T}) * |W(e^{j\omega T})|^2$$

This result shows that the modified periodogram introduces frequency domain smoothing (convolution) dependent upon the window chosen and the data length. As for

(cont.

the standard periodogram, the smoothed estimate is guaranteed to be positive. Choice of better windows (hanning, hamming, ...) than the rectangular window used for standard periodograms will give other trade-offs between side-lobe heights (spectral leakage) and main lobe width (smearing) which may be more appropriate for frequency resolution of more than one frequency component and smoothing of noise-based effects. The resolution will increase in proportion to the data length for a constant shape of window function.

4 (a) **Parametric methods**

- Periodogram-based methods can lead to biased estimators with large variance
- If the physical process which generated the data is known or can be well approximated, then a parametric model can be constructed
- Careful estimation of the parameters in the model can lead to power spectrum estimates with improved bias/variance.
- We will consider spectrum estimation for LTI systems driven by a white noise input sequence.
- If a random process $\{X_n\}$ can be modelled as white noise exciting a filter with frequency response $H(e^{j\omega T})$ then the spectral density of the data can be expressed as:

$$\boxed{S_X(e^{j\omega T}) = \sigma_w^2 |H(e^{j\omega T})|^2}$$

where σ_w^2 is the variance of the white noise process. [It is usually assumed that $\sigma_w^2 = 1$ and the scaling is incorporated as gain in the frequency response]

- We will study models in which the frequency response $H(e^{j\omega T})$ can be represented by a finite number of parameters which are estimated from the data.
- Parametric models need to be chosen carefully - an inappropriate model for the data can give misleading results

ARMA Models A quite general representation is the autoregressive moving-average (ARMA) model:

- The ARMA(P,Q) model difference equation representation is:

$$\boxed{x_n = -\sum_{p=1}^{P} a_p x_{n-p} + \sum_{q=0}^{Q} b_q w_{n-q}} \qquad (8)$$

(cont.

where:

a_p are the AR parameters,

b_q are the MA parameters

and $\{W_n\}$ is a zero-mean stationary white noise process with unit variance, $\sigma_w^2 = 1$.

•Clearly the ARMA model is a pole-zero IIR filter-based model with transfer function

$$H(z) = \frac{B(z)}{A(z)}$$

where:

$$A(z) = 1 + \sum_{p=1}^{P} a_p z^{-p}, \quad B(z) = \sum_{q=0}^{Q} b_q z^{-q}$$

•Unless otherwise stated we will always assume that the filter is stable, i.e. the poles (solutions of $A(z) = 0$) all lie *within* the unit circle (we say in this case that $A(z)$ is *minimum phase*). Otherwise the autocorrelation function is undefined and the process is technically *non-stationary*.

•Hence the power spectrum of the ARMA process is:

$$S_X(e^{j\omega T}) = \frac{|B(e^{j\omega T})|^2}{|A(e^{j\omega T})|^2}$$

The ARMA model is quite a flexible and general way to model a stationary random process:

•The poles model well the *peaks* in the spectrum (sharper peaks implies poles closer to the unit circle)

•The zeros model troughs in the spectrum

•Complex spectra can be approximated well by large model orders P and Q

[This is more detailed than required]

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(b) The AR model can be written as:

$$x_n = -\sum_{p=1}^{P} a_p x_{n-p} + e_n \tag{9}$$

where e_n is a white noise sequence having variance $\sigma_e^2 = b_0^2$.

An alternative interpretation of this equation is that:

$$x_n = \hat{x}_n + e_n$$

where:

$$\hat{x}_n = -\sum_{p=1}^{P} a_p x_{n-p}$$

is a prediction of x_n from previous data and the term e_n is the *prediction error*. In terms of e_n, equation 9 becomes:

$$e_n = x_n + \sum_{p=1}^{P} a_p x_{n-p}$$

Suppose we write this equation for all values of n such that

$$n_I \le n \le n_F$$

All of these equations may be expressed in matrix notation as:

$$\boxed{\mathbf{e} = \mathbf{x} + \mathbf{X}\mathbf{a}}$$

where:

$$\mathbf{e} = \begin{bmatrix} e_{n_I} \\ e_{n_I+1} \\ \vdots \\ e_{n_F} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_{n_I} \\ x_{n_I+1} \\ \vdots \\ x_{n_F} \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} x_{n_I-1} & x_{n_I-2} & \cdots & x_{n_I-P} \\ x_{n_I} & x_{n_I-1} & \cdots & x_{n_I-P+1} \\ \vdots & \vdots & & \vdots \\ x_{n_F-1} & x_{n_F-2} & \cdots & x_{n_F-P} \end{bmatrix} \tag{10}$$

and

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_P \end{bmatrix}$$

(cont.

If we wish to find the AR parameters which fit the observed data 'best', then it would seem reasonable to minimize the prediction error terms (i.e. an 'ideal' model for the data would have zero prediction errors).

A convenient way to achieve this by choosing the parameter vector $\mathbf{a}$ which minimizes the total squared prediction error, E:

$$\mathscr{E} = \sum_{n=n_I}^{n_F} e_n^2 = \mathbf{e}^{\mathrm{T}}\mathbf{e}$$

where $\mathbf{e}^{\mathrm{T}}$ denotes the transpose of $\mathbf{e}$.

We recognise this as a standard least squares estimation problem, as studied in E4 Linear Algebra, so we obtain the solution immediately:

$$\boxed{\mathbf{a} = -(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{x}}$$

An alternative derivation seeks to minimize the function directly. In order to find the minimum of $\mathbf{e}^{\mathrm{T}}\mathbf{e}$ with respect to all of the elements of $\mathbf{a}$ we must solve the P simultaneous equations:

$$\frac{\partial(\mathbf{e}^{\mathrm{T}}\mathbf{e})}{\partial a_i} = 0,\ i = 1,2,...,P$$

We can express the same thing in vector notation as:

$$\frac{\partial(\mathbf{e}^{\mathrm{T}}\mathbf{e})}{\partial \mathbf{a}} = \mathbf{0}_P$$

where

$$\frac{\partial(\mathbf{e}^{\mathrm{T}}\mathbf{e})}{\partial \mathbf{a}} = \begin{bmatrix} \frac{\partial(\mathbf{e}^{\mathrm{T}}\mathbf{e})}{\partial a_1} \\ \frac{\partial(\mathbf{e}^{\mathrm{T}}\mathbf{e})}{\partial a_2} \\ \vdots \\ \frac{\partial(\mathbf{e}^{\mathrm{T}}\mathbf{e})}{\partial a_P} \end{bmatrix} \quad \text{and} \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

[so $\frac{\partial(\mathbf{e}^{\mathrm{T}}\mathbf{e})}{\partial \mathbf{a}}$ is just the *gradient vector* from vector calculus].

Now, expand $\mathbf{e}^{\mathrm{T}}\mathbf{e}$ and differentiate:

$$\mathbf{e} = \mathbf{x} + \mathbf{X}\mathbf{a}$$

$$\mathbf{e}^{\mathrm{T}}\mathbf{e} = (\mathbf{x} + \mathbf{X}\mathbf{a})^{\mathrm{T}}(\mathbf{x} + \mathbf{X}\mathbf{a})$$

$$= \mathbf{x}^{\mathrm{T}}\mathbf{x} + 2\mathbf{x}^{\mathrm{T}}\mathbf{X}\mathbf{a} + \mathbf{a}^{\mathrm{T}}\mathbf{X}^{\mathrm{T}}\mathbf{X}\mathbf{a}$$

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$$\frac{\partial(\mathbf{e}^{\mathrm{T}}\mathbf{e})}{\partial\mathbf{a}} = 2\frac{\partial(\mathbf{x}^{\mathrm{T}}\mathbf{X}\mathbf{a})}{\partial\mathbf{a}} + \frac{\partial(\mathbf{a}^{\mathrm{T}}\mathbf{X}^{\mathrm{T}}\mathbf{X}\mathbf{a})}{\partial\mathbf{a}}$$
$$= 2\mathbf{X}^{\mathrm{T}}\mathbf{x} + 2\mathbf{X}^{\mathrm{T}}\mathbf{X}\mathbf{a}$$

Here we have used two standard results from matrix/vector calculus:

$$\frac{\partial(\mathbf{b}^{\mathrm{T}}\mathbf{a})}{\partial\mathbf{a}} = \frac{\partial(\mathbf{a}^{\mathrm{T}}\mathbf{b})}{\partial\mathbf{a}} = \mathbf{b} \text{ and } \frac{\partial(\mathbf{a}^{\mathrm{T}}\mathbf{B}\mathbf{a})}{\partial\mathbf{a}} = 2\mathbf{B}\mathbf{a}$$

for constant vector $\mathbf{b}$ and symmetric matrix $\mathbf{B}$.

[You can verify these by differention term by term. See, for example Therrien, Appendix A.]

For a maximum or minimum of $\mathbf{e}^{\mathrm{T}}\mathbf{e}$:

$$\frac{\partial(\mathbf{e}^{\mathrm{T}}\mathbf{e})}{\partial\mathbf{a}} = \mathbf{0}$$

Therefore,

$$2\mathbf{X}^{\mathrm{T}}\mathbf{x} + 2\mathbf{X}^{\mathrm{T}}\mathbf{X}\mathbf{a} = \mathbf{0} \tag{11}$$

and finally, provided $\mathbf{X}$ is full rank,

$$\boxed{\mathbf{a} = -(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{x}}$$

Covariance method

- The covariance method minimizes only those error terms which can be fully calculated from the data

- Examine the error equation:

$$e_n = x_n + \sum_{p=1}^{P} a_p x_{n-p}$$

- The first error term that can be fully calculated is e_P and the last is e_{N-1}.

- Hence $n_I = P$ and $n_F = N-1$ in the squared error equation:

$$\mathscr{E}^{\mathrm{C}} = \sum_{n=P}^{N-1} e_n^2$$

(cont.

- The resulting matrix $\mathbf{X}^T\mathbf{X}$ is *not* Toeplitz. Although fast algorithms exist to solve for $\mathbf{a}$, they are much more complex than for the autocorrelation method.

- The AR parameter estimate is not guaranteed to be stable

- The method is intuitively appealing as it does not attempt to make guesses about data that aren't observed.

- The covariance method is a good approximation for moderately large N to the true maximum likelihood estimate.

Autocorrelation method

- In the autocorrelation method $n_I = 0$ and $n_F = N + P - 1$

- Hence the squared error minimized is:

$$\mathcal{E}^{\mathrm{A}} = \sum_{n=0}^{N+P-1} e_n^2$$

- To calculate these error terms requires data before $n = 0$ and after $n = N - 1$. These data points are assumed to be zero.

- $\mathbf{X}^T\mathbf{X}$ is *Toeplitz*, which means that the efficient Levinson recursion ($O(P^2)$) can be used to solve for $\mathbf{a}$.

- Note that the autocorrelation method is equivalent to estimating the autocorrelation function using the unbiased estimate and then solving the matrix Yule-Walker equations directly

- The parameter estimate is guaranteed to be stable

- However, the assumption of zeros before the start and after the end of the data are likely to make the estimate less accurate than the covariance method for small N

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(c) For the covariance method we have:

$$\mathbf{x} = [\beta, \beta^2, ..., \beta^{N-1}]^T$$

and

$$\mathbf{X} = [1, \beta, \beta^2, ..., \beta^{N-2}]^T$$

Hence:

$$\mathbf{X}^T\mathbf{X} = \sum_{n=0}^{N-2} \beta^{2n}$$

and

$$\mathbf{X}^T\mathbf{x} = \sum_{n=0}^{N-2} \beta^n \beta^{n+1} = \beta \sum_{n=0}^{N-2} \beta^{2n}$$

[N.B. no need to sum these GPs directly]

Finally,

$$\mathbf{a} = a_1 = -(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{x} = -\beta$$

The prediction error is

$$e_n = x_n + a_1 x_{n-1}$$

which is zero for $n = 1, 2, ..., N-1$. Thus the model fits the data perfectly in this case, with zero error.

The transfer function of the all-pole filter is:

$$H(z) = \frac{1}{1 + a_1 z^{-1}}$$

so the pole is at $z = -a_1 = \beta$. Thus the estimated filter is only unconditionally stable for $|\beta| < 1$ and hence only gives a valid power spectrum estimate for that range of values. [On the other hand, with $|\beta| > 1$ the model that generated the data is likely to be an unstable model, so we shouldn't see this as too much of a problem in this particular idealised case].

END OF PAPER