1. (i) $$w = a\sin z = \frac{a}{2i}\left(e^{iz} - e^{-iz}\right)$$

$$\Rightarrow \frac{2iw}{a} = e^{iz} - e^{-iz}$$

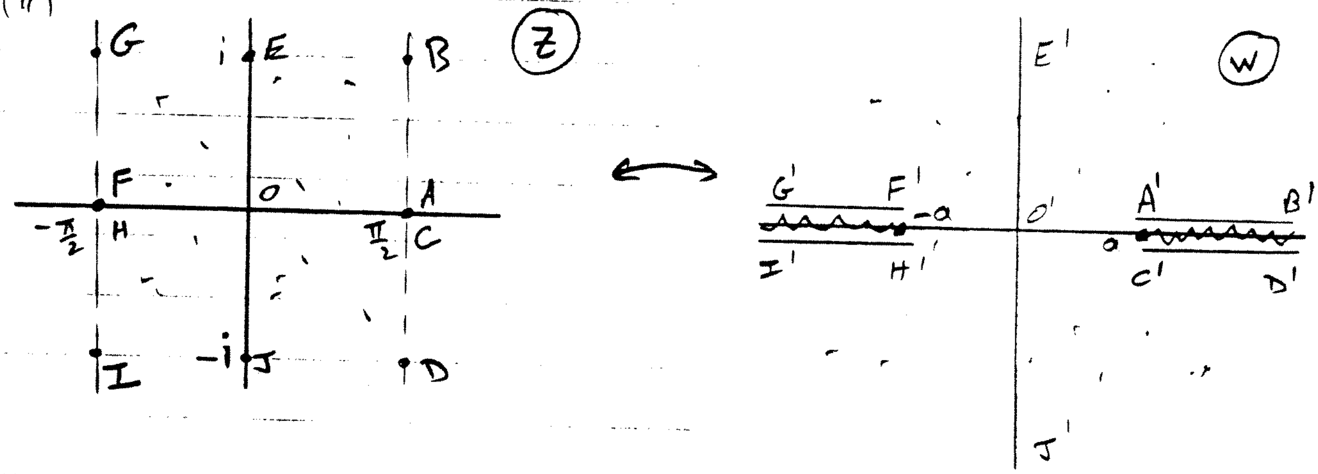
$$\Rightarrow e^{i2z} - \frac{2iw}{a}e^{iz} = 1$$

$$\Rightarrow \left(e^{iz} - \frac{iw}{a}\right)^2 = 1 - \left(\frac{w}{a}\right)^2$$

$$\Rightarrow e^{iz} = \frac{iw}{a} + \left[1 - \left(\frac{w}{a}\right)^2\right]^{1/2}$$

$$\Rightarrow iz = \ln\left\{\frac{iw}{a} + \left[1 - \left(\frac{w}{a}\right)^2\right]^{1/2}\right\}$$ [20%]

(ii)



2 Branch cuts exist in the mapped w-plane.

Proof

Line $O'A'$: $z = x + iy$ with $y = 0$, $0 < x < \frac{\pi}{2}$

$\Rightarrow$ $w = a\sin x$

Line $O'F'$: again $w = a\sin x$, with $-\frac{\pi}{2} < x = 0$

$A'B'$: $z = \frac{\pi}{2} + iy \Rightarrow w = a\sin\left(\frac{\pi}{2} + iy\right) = a\cos iy = a\cosh y$

$C'D'$: $z = \frac{\pi}{2} - iy \Rightarrow w = a\cosh y$

$F'G'$: $z = -\frac{\pi}{2} + iy \Rightarrow w = a\sin\left(-\frac{\pi}{2} + iy\right) = -a\cosh y$

$H'I'$: $z = -\frac{\pi}{2} - iy \Rightarrow w = -a\cosh y$

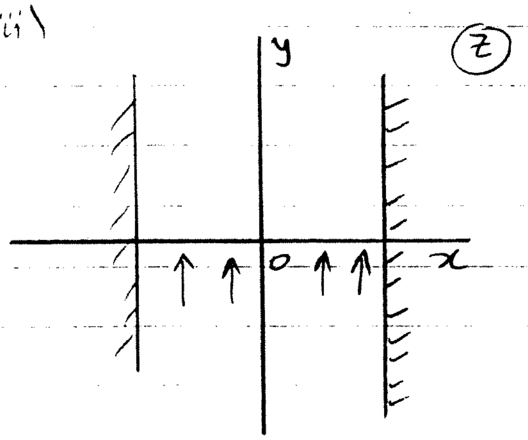
1. (ii) contd.

$O'E'$: $z = iy \Rightarrow w = a \sin iy = i a \sinh y$

$O'J'$: $z = -iy \Rightarrow W = i a \sinh y$.

[4 0l.]

(iii)



In z-domain, $\nabla^2\phi = \nabla^2\psi = 0$ where

$$\Omega(z) = \phi(x,y) + i\psi(x,y)$$

Arrange for uniform flow in the y-direction : try $\Omega(z) = -icz$

Then $v_x - i v_y = \Omega'(z) = -ic \Rightarrow \underline{v_y = c}$

Stream function $\psi = ?$

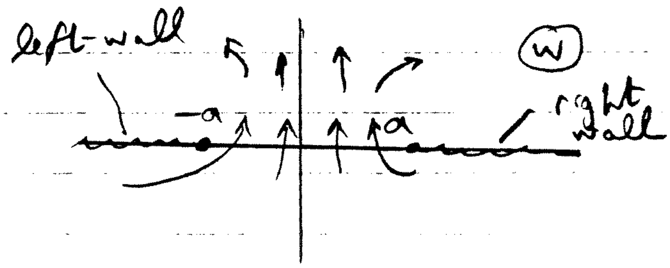
$$\Omega(z) = -icz = -ic(x+iy) = cy - icx$$
$$\Rightarrow \psi(x,y) = -cx$$

$$\Rightarrow \psi = c\frac{\pi}{2} \text{ along } x = -\pi/2$$
$$= -c\frac{\pi}{2} \text{ along } x = \pi/2.$$

So, in mapped w-plane, we have

$$\Omega(z) = \Omega[z(w)] = -ciz$$
$$\Rightarrow \Omega = -c \ln\left[\frac{iw}{a} + \left(1 - \left(\frac{w}{a}\right)^2\right)^{1/2}\right]$$

with $\psi = c\pi/2$ along the left wall
$= -c\pi/2$ .. .. right wall



$\Delta\psi = c\pi = Q$

$\Rightarrow \underline{c = Q/\pi}$

[4 0l.

2. (a) $I = \int_0^\infty \frac{x^2}{(x^2+4)^2} = ?$ Consider $J = \oint_C \frac{z^2}{(z^2+4)^2}\,dz$

Now, $(z^2+4)^2 = (z+2i)^2(z-2i)^2$ so there are poles of order 2 at $z = \pm 2i$.

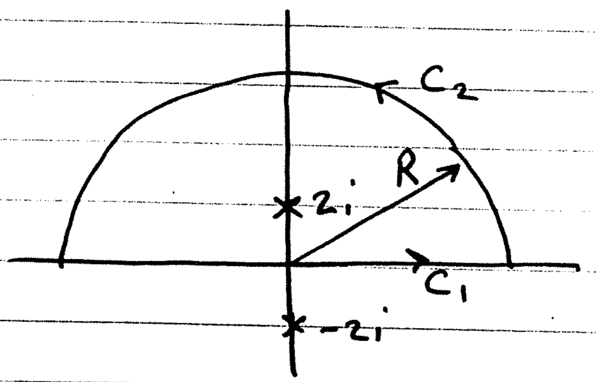
$$f(z) = \frac{z^2}{(z^2+4)^2} = \frac{z^2}{(z+2i)^2(z-2i)^2}$$

Residue at $z = 2i$ is given by

$$\text{Res} = \lim_{z \to 2i} \frac{d}{dz}\left[(z-2i)^2 f(z)\right]$$

$$= \lim_{z \to 2i} \frac{d}{dz}\left[\frac{z^2}{(z+2i)^2}\right] = \frac{1}{8i}.$$

Consider the closed contour $C = C_1 + C_2$,



$$J = 2I + \int_{C_2} \frac{z^2}{(z^2+4)^2}\,dz$$

Integral on C_2 vanishes as $R \to \infty$.

So, $J = 2I = 2\pi i \times$ Residue at $z = 2i$

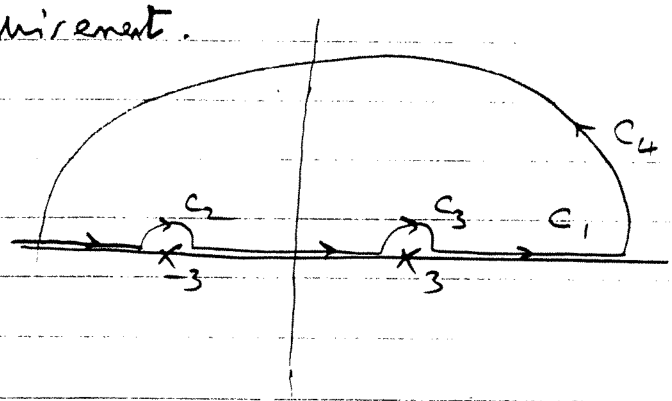
$\Rightarrow$ $I = \pi i \times$ Residue $\Rightarrow$ $\underline{\underline{I = \pi/8}}$

(50%)

2. (b) $I = \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2-9}\, dx$

Consider $J = \oint_C \frac{e^{iaz}}{z^2-9}\, dz$ with $\underline{a>0}$.

The integrand $f(z) = \frac{e^{iaz}}{z^2-9}$ has simple poles at $z = \pm 3$, hence the Principal Value requirement.



$$I = \int_{C_1} \frac{e^{iaz}}{z^2-9}\, dz = ? \qquad J = \oint_C \frac{e^{iaz}}{z^2-9}\, dz = 0$$

Residue at $z = 3$ is ?

$$f(z) = \frac{e^{iaz}}{(z-3)(z+3)} \Rightarrow \underset{z\to 3}{\text{residue}} = \frac{e^{i3a}}{6}$$

$$\Rightarrow \int_{C_3} f(z)\, dz = -\pi i \cdot \frac{e^{i3a}}{6}$$

Residue at $z = -3$ is ?

$$\underset{z\to -3}{\text{Residue}} = \frac{e^{-i3a}}{-6} \Rightarrow \int_{C_2} f(z)\, dz = -\pi i \cdot \frac{e^{-i3a}}{-6}$$

$\int_{C_4} f(z)\, dz = 0$ for $a>0$ by Jordan's Lemma.

Hence, $J = I + \int_{C_2+C_3} f(z)\, dz = 0$

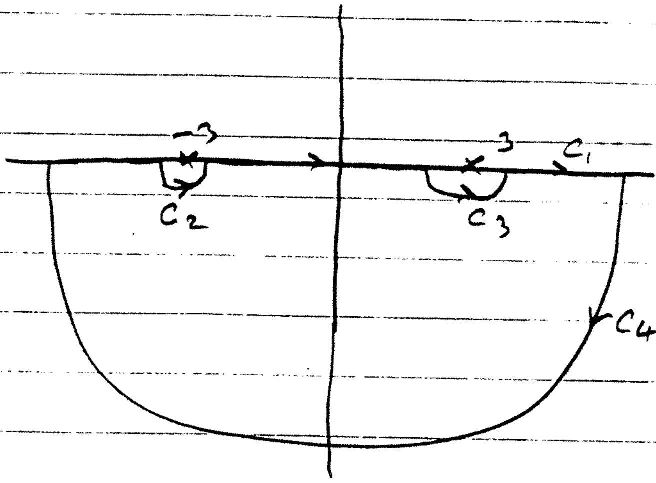
$$\Rightarrow I = \frac{\pi i}{6} e^{i3a} - \frac{\pi i}{6} e^{-i3a} = \frac{\pi}{6} i \left(e^{i3a} - e^{-i3a}\right)$$

2. (b) contd.

$$e^{i3a} - e^{-i3a} = 2i \sin 3a$$

$$\Rightarrow \quad I = -\frac{\pi}{3} \sin 3a \quad , \quad a > 0$$

Now consider $a < 0$.



$$J = \oint_C f(z)\, dz = 0$$

$\int_{C_4} f(z)\, dz = 0$ by Jordan's lemma.

$$\int_{C_2} f(z)\, dz = -\frac{\pi i}{6} e^{-i3a}$$

$$\int_{C_3} f(z)\, dz = \frac{\pi i}{6} e^{i3a}$$

So $\quad J = 0 = I + \int_{C_2 + C_3} f(z)\, dz$

$$\Rightarrow \quad I = -\int_{C_2} f(z)\, dz - \int_{C_3} f(z)\, dz$$

$$= \frac{\pi i}{6}\left(e^{-i3a} - e^{+i3a}\right) = +\frac{\pi}{3} \sin 3a$$

So $\quad I = +\frac{\pi}{3} \sin 3a \quad , \quad a < 0.$

So $\quad I = -\frac{\pi}{3} \sin |3a| \quad$ for all real a.

[50%]

3 (a) $$\delta \int (T-V)\,dt = \int \left\{ \frac{\partial T}{\partial \dot q_1}\delta \dot q_1 + \frac{\partial T}{\partial \dot q_2}\delta \dot q_2 + \frac{\partial T}{\partial q_1}\delta q_1 + \frac{\partial T}{\partial q_2}\delta q_2 - \frac{\partial V}{\partial q_1}\delta q_1 - \frac{\partial V}{\partial q_2}\delta q_2 \right\} dt$$

integrate by parts to give $-\frac{d}{dt}\left[\frac{\partial T}{\partial \dot q_1}\right]\delta q_1$

repeat for term involving $\delta \dot q_2$

Thus $$\delta \int (T-U)\,dt = \int \left\{ \delta q_1 \left[-\frac{d}{dt}\left[\frac{\partial T}{\partial \dot q_1}\right] + \frac{\partial T}{\partial q_1} - \frac{\partial V}{\partial q_1} \right] + \delta q_2 \left[-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_2}\right) + \frac{\partial T}{\partial q_2} - \frac{\partial V}{\partial q_2} \right] \right\} dt$$

+ boundary terms

True for all δq_1 and δq_2 implies that

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{\partial T}{\partial \dot q_1}\right] - \frac{\partial T}{\partial q_1} + \frac{\partial V}{\partial q_1} &= 0 \\ \frac{d}{dt}\left[\frac{\partial T}{\partial \dot q_2}\right] - \frac{\partial T}{\partial q_2} + \frac{\partial V}{\partial q_2} &= 0 \end{aligned}\right\} \text{Lagrange's equations}$$

[30%]

(b) The variational statement must be modified :-

$$\delta \left\{ \int (T-U)\,dt + \int \underset{\substack{\uparrow \\ \text{Lagrange Multiplyer}}}{\lambda} C \,dt \right\} = 0$$

Taking the variation, the following <u>additional</u> terms appear :-

$$\int \Big[\underbrace{\delta\lambda . C}_{\substack{\downarrow \\ \text{just gives } C=0 \text{, the original constraint equation}}} + \underbrace{\lambda \frac{\partial C}{\partial q_1}\delta q_1 + \lambda \frac{\partial C}{\partial q_2}\delta q_2}_{\text{produce additional terms in equations of motion}} \Big] dt$$

Eqns of motion become:

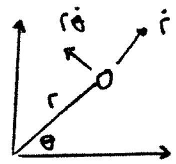
$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_1}\right) - \frac{\partial T}{\partial q_1} + \frac{\partial V}{\partial q_1} - \lambda \frac{\partial C}{\partial q_1} &= 0 \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_2}\right) - \frac{\partial T}{\partial q_2} + \frac{\partial V}{\partial q_2} - \lambda \frac{\partial C}{\partial q_2} &= 0 \end{aligned}\right\} \text{together with } C=0.$$

[35%]

(c) $T = \frac{1}{2}m\,(\dot{r}^2 + r^2\dot{\theta}^2)$

$V = \frac{1}{2}k\,(r\cos\theta)^2$

$C = r\sin\theta - a$



$r \equiv q_1$

$\theta \equiv q_2$

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial T}{\partial r} = Mr\dot{\theta}^2 \qquad \frac{\partial V}{\partial r} = kr\cos^2\theta \qquad \frac{\partial C}{\partial r} = \sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} + \frac{\partial V}{\partial r} - \lambda\frac{\partial C}{\partial r} = 0 \quad \Rightarrow \quad \underline{M\ddot{r} - Mr\dot{\theta}^2 + kr\cos^2\theta - \lambda\sin\theta = 0} \qquad \text{— (1)}$$

$$\frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \frac{\partial T}{\partial \theta} = 0 \qquad \frac{\partial V}{\partial \theta} = -kr^2\cos\theta\sin\theta \qquad \frac{\partial C}{\partial \theta} = r\cos\theta$$

$$\Rightarrow \quad \underline{M\frac{d}{dt}(r^2\dot{\theta}) - kr^2\cos\theta\sin\theta - \lambda r\cos\theta = 0} \qquad \text{— (2)}$$

(1) × $r\cos\theta$ − (2) × $\sin\theta$, to eliminate λ:-

$$(M\ddot{r} - Mr\dot{\theta}^2)\,r\cos\theta - M\frac{d}{dt}(r^2\dot{\theta})\sin\theta + kr^2\cos^3\theta + kr^2\cos\theta\sin^2\theta = 0$$

$$\underbrace{(M\ddot{r} - Mr\dot{\theta}^2)\cos\theta - 2M\dot{r}\dot{\theta}\sin\theta - Mr\ddot{\theta}\sin\theta}_{} + kr\cos\theta = 0$$

$$= M\frac{d}{dt}\left[\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right] = M\frac{d^2}{dt^2}(r\cos\theta)$$

$$\Rightarrow \quad M\frac{d^2}{dt^2}(r\cos\theta) + k\,(r\cos\theta) = 0 \quad ; \quad \text{Put } x = r\cos\theta \quad \Rightarrow \quad \underline{M\ddot{x} + kx = 0}$$ [35%]

6 (a)

$$L = \frac{1}{2\rho_0 c^2} \int_V [p^2 - (c/\omega)^2 \nabla p . \nabla p] \, dV + \frac{1}{2} \int_S [T \nabla u . \nabla u - \omega^2 m u^2 - 2pu] \, ds$$

$$\Rightarrow \delta L = \frac{1}{\rho_0 c^2} \int_V [p \, \delta p - (c/\omega)^2 \underbrace{\nabla p . \nabla \delta p}] \, dV + \int_S [T \nabla u . \nabla \delta u - \omega^2 m \, \delta u \, u - p \, \delta u - u \, \delta p] \, ds$$

consider $\nabla p . \nabla \delta p = \frac{\partial p}{\partial x_i} \frac{\partial \delta p}{\partial x_i} = \frac{\partial}{\partial x_i}[\delta p . \frac{\partial p}{\partial x_i}] - \delta p \frac{\partial^2 p}{\partial x_i^2}$

$$\Rightarrow \int \nabla p . \nabla \delta p \, dV = \int_S \delta p \frac{\partial p}{\partial x_i} n_i \, ds - \int_V \delta p \frac{\partial^2 p}{\partial x_i^2} \, dV$$

$$= \int_S \delta p \, \nabla p . \underset{\sim}{n} \, ds - \int_V \delta p \, \nabla^2 p \, dV$$

Similarly $\int_S \nabla u . \nabla \delta u \, ds = - \int_S \delta u . \nabla^2 u \, ds$

So, collecting terms in the integrals :-

$$\delta L = \frac{1}{\rho_0 c^2} \int_V \delta p . [p + (c/\omega)^2 \nabla^2 p] \, dV + \int_S \{ \delta u [-\omega^2 m u - T \nabla^2 u - p] + \delta p [-u + (\frac{-1}{\rho_0 \omega^2}) \nabla p . \underset{\sim}{n}] \} \, ds$$

from volume integral

Thus: $\underline{p + (c/\omega)^2 \nabla^2 p = 0 \quad \text{in } V}$ [20%]

$\underline{-\omega^2 m u - T \nabla^2 u = p \quad \text{on } S}$ [20%]

$\underline{\nabla p . \underset{\sim}{n} = -\rho_0 \omega^2 u \quad \text{on } S}$ [20%]

(b) $\underset{\sim}{a} \cdot \operatorname{curl}(\underset{\sim}{b} \times \underset{\sim}{x}) = a_i \, \varepsilon_{ijk} \frac{\partial}{\partial x_j} (\underset{\sim}{b} \times \underset{\sim}{x})_k$

$$= a_i \, \varepsilon_{ijk} \frac{\partial}{\partial x_j} \varepsilon_{krs} b_r x_s = a_i \underbrace{\varepsilon_{ijk} \varepsilon_{krs}} \left(x_s \frac{\partial b_r}{\partial x_j} + b_r \delta_{js} \right)$$

$$\varepsilon_{ijk} \varepsilon_{krs} = \varepsilon_{kij} \varepsilon_{krs}$$

$$= \delta_{ir} \delta_{js} - \delta_{is} \delta_{jr}$$

So $\underset{\sim}{a} \cdot \operatorname{curl}(\underset{\sim}{b} \times \underset{\sim}{x}) = a_i (\delta_{ir} \delta_{js} - \delta_{is} \delta_{jr}) \left(x_s \frac{\partial b_r}{\partial x_j} + b_r \delta_{js} \right)$

$$= a_i \frac{\partial b_i}{\partial x_j} x_j + a_i b_i \delta_{jj} - a_i x_i \frac{\partial b_j}{\partial x_j} - a_i b_i$$

$$= \underset{\sim}{a} \cdot [(\underset{\sim}{x} \cdot \nabla) \underset{\sim}{b}] + 3 \underset{\sim}{a} \cdot \underset{\sim}{b} - (\underset{\sim}{a} \cdot \underset{\sim}{x}) \operatorname{div}(\underset{\sim}{b}) - \underset{\sim}{a} \cdot \underset{\sim}{b}$$

$$= \underline{2 \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot [(\underset{\sim}{x} \cdot \nabla) \underset{\sim}{b}] - (\underset{\sim}{a} \cdot \underset{\sim}{x}) \operatorname{div}(\underset{\sim}{b})}$$ [40%]