1)

i) Column space

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 8 \\ -1 \end{pmatrix} \quad \text{for 2 pivots}$$

ii) Row space

$$\begin{pmatrix} 1 \\ 3 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -4 \\ 2 \end{pmatrix}$$

iii) Null space

Solve $Ux = 0$

setting each free variable in turn to 1

$$\begin{pmatrix} 1 & 3 & -1 & -2 \\ 0 & 2 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -5 \\ 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -5 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

1.cont.

iv) Left Null space

$$L L^{-1} = I$$

$$\Longrightarrow \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}$$

c) $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 3 & -1 & -2 \\ 0 & 2 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Number of pivots $= 2$.

$\therefore$ rank $= 2$.

2/

a) Bookwork..

b) Let S be the space of solutions to the equation

$$x_1 - x_2 + x_3 - 2x_4 = 0$$

$$\begin{pmatrix} 1 & -1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0.$$

Thus we find a basis for the Null space of

$$A = \begin{pmatrix} 1 & -1 & 1 & -2 \end{pmatrix}$$

Col 2 is a multiple of Col 1

$$\text{Col } 2 = -1 \cdot \text{Col } 1$$

$\therefore$ $1 \cdot \text{col } 1 + 1 \cdot \text{col } 2 + 0 \cdot \text{col } 3 + 0 \cdot \text{col } 4 = 0$

$\therefore$ a basis element of $N(A)$ is $a = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

2 cont.

Similarly, col 3 = 1 . col 1

$\therefore b = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ is a second basis

Finally, col 4 = − 2 . col 1

$\therefore c = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ is a final basis

$$\therefore B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for the null space of A.

2 cont.

c) Let $A = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$$B = b - \frac{b.A}{A.A} A = \begin{pmatrix} -1/2 \\ 1/2 \\ 1 \\ 0 \end{pmatrix}$$

$$C = c - \frac{c.A}{A.A} A - \frac{c.B}{B.B} B$$

$$= \begin{pmatrix} 2/3 \\ -2/3 \\ 2/3 \\ 1 \end{pmatrix}$$

$$\Rightarrow q_A = \frac{A}{\|A\|} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, \quad q_B = \frac{B}{\|B\|} = \begin{pmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \\ 0 \end{pmatrix}, \quad q_c = \frac{C}{\|C\|} = \begin{pmatrix} 2/\sqrt{21} \\ -2/\sqrt{21} \\ 2/\sqrt{21} \\ 3/\sqrt{21} \end{pmatrix}$$

~~2 cont~~

d) Project b onto S

Projection matrix $P = Q(Q^TQ)^{-1}Q^T$

$= QQ^T$

$$Q = (q_A, q_B, q_C) = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 2/\sqrt{21} \\ 1/\sqrt{2} & 1/\sqrt{6} & -2/\sqrt{21} \\ 0 & 2/\sqrt{6} & 2/\sqrt{21} \\ 0 & 0 & 3/\sqrt{21} \end{pmatrix}$$

$$\therefore P = \frac{1}{7}\begin{pmatrix} 6 & 1 & -1 & 2 \\ 1 & 6 & 1 & -2 \\ -1 & 1 & 6 & 2 \\ 2 & -2 & 2 & 3 \end{pmatrix}$$

$\therefore$ Vector closest to $b = (1\ 1\ 1\ 1)^T$

is

$$QQ^Tb = \frac{1}{7}\begin{pmatrix} 6 & 1 & -1 & 2 \\ 1 & 6 & 1 & -2 \\ -1 & 1 & 6 & 2 \\ 2 & -2 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 8/7 \\ 6/7 \\ 8/7 \\ 5/7 \end{pmatrix}$$

3 (a) Penalty functions penalise solutions for violating constraints by adding a penalty term to the raw objective function, if constraints are violated.

Barrier functions penalise solutions for approaching the edge of feasible space.

The advantages of penalty functions are:

1. They can handle both equality and inequality constraints
2. The search can start from an arbitrary solution.
3. The search can iterate through infeasible space (essential if feasible space is disjoint).

The disadvantage of penalty functions is:

1. The search can get stuck in infeasible space.

The advantage of barrier functions is:

1. It is guaranteed to yield a feasible solution.

The disadvantages of barrier functions are:

1. They can only handle inequality constraints.
2. They need a feasible starting point (not always easy to find).
3. The search cannot iterate through infeasible space.

(b) The term $4xz + 4x^2$ represents the raw objective function (the surface area of material used).

The standard penalty term for an inequality constraint $g(\underline{x}) \leq 0$ is $p(\max[0, g(\underline{x})])^2$.

To handle an equality constraint $h(\underline{x}) = 0$ using penalty functions it is converted to two inequality constraints $h(\underline{x}) \leq 0$ and $-h(\underline{x}) \leq 0$.

Thus the penalty term becomes

$$p\{(\max[0, h(\underline{x})])^2 + (\max[0, -h(\underline{x})])^2\}$$

which is simply equal to $p(h(\underline{x}))^2 = p(x^2z - V)^2$

3 (c) $f = 4xz + 4x^2 + \rho(x^2 z - V)^2$

$$\therefore \frac{\partial f}{\partial x} = 4z + 8x + 4\rho xz(x^2 z - V)$$

$$\frac{\partial f}{\partial z} = 4x + 2\rho x^2(x^2 z - V)$$

$$\frac{\partial^2 f}{\partial x^2} = 8 + 4\rho z(x^2 z - V) + 8\rho x^2 z^2$$
$$= 8 + 12\rho x^2 z^2 - 4\rho z V$$

$$\frac{\partial^2 f}{\partial z^2} = 2\rho x^4$$

$$\frac{\partial^2 f}{\partial x \partial z} = 4 + 4\rho x(x^2 z - V) + 4\rho x^3 z$$
$$= 4 + 8\rho x^3 z - 4\rho x V$$

For $V = 2$, $\rho = 5$ and $x_0 = 1$, $z_0 = 1$

$$\frac{\partial f}{\partial x} = -8 \qquad \frac{\partial f}{\partial z} = -6$$

$$\frac{\partial^2 f}{\partial x^2} = 28 \qquad \frac{\partial^2 f}{\partial z^2} = 10 \qquad \frac{\partial^2 f}{\partial x \partial z} = 4$$

Hence $\underline{\underline{H}} = \begin{bmatrix} 28 & 4 \\ 4 & 10 \end{bmatrix} \Rightarrow \underline{\underline{H}}^{-1} = \frac{1}{264}\begin{bmatrix} 10 & -4 \\ -4 & 28 \end{bmatrix}$

For Newton's Method $\underline{x}_{k+1} = \underline{x}_k - \underline{\underline{H}}^{-1} \underline{\nabla f}(\underline{x}_k)$

$$\therefore \underline{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{264}\begin{bmatrix} 10 & -4 \\ -4 & 28 \end{bmatrix}\begin{bmatrix} -8 \\ -6 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1.212 \\ 1.515 \end{bmatrix}}}$$

For $x_1 = 1.212$, $z_1 = 1.515$

$$\frac{\partial f}{\partial x} = 24.06 \qquad \frac{\partial f}{\partial z} = 8.17$$

$$\frac{\partial^2 f}{\partial x^2} = 149.77 \qquad \frac{\partial^2 f}{\partial z^2} = 21.59 \qquad \frac{\partial^2 f}{\partial x \partial z} = 63.45$$

$$\therefore \underline{x}_2 = \begin{bmatrix} 1.212 \\ 1.515 \end{bmatrix} - \frac{1}{-792.37}\begin{bmatrix} 21.59 & -63.45 \\ -63.45 & 149.77 \end{bmatrix}\begin{bmatrix} 24.06 \\ 8.17 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1.213 \\ 1.133 \end{bmatrix}}}$$

3 (c) continued

For $x_2 = 1.213$, $z_2 = 1.133$

$$\frac{\partial f}{\partial x} = 5.110 \qquad \frac{\partial f}{\partial z} = -0.035$$

$$\frac{\partial^2 f}{\partial x^2} = 76.06 \qquad \frac{\partial^2 f}{\partial z^2} = 21.68 \qquad \frac{\partial^2 f}{\partial x \partial z} = 36.42$$

$$\therefore \underline{x}_3 = \begin{bmatrix} 1.213 \\ 1.133 \end{bmatrix} - \frac{1}{322.39}\begin{bmatrix} 21.68 & -36.42 \\ -36.42 & 76.06 \end{bmatrix}\begin{bmatrix} 5.110 \\ -0.035 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 0.866 \\ 1.718 \end{bmatrix}}}$$

Newton's Method appears to be converging on the neighbourhood of the global optimum ($x = 1$, $z = 2$). However, it will only locate this solution accurately for a very large value of the penalty parameter p. (In fact, for $p = 5$ Newton's Method converges on $x = 0.922$, $z = 1.843$.)

(d) Difficulties can occur if the function to be optimised is poorly approximated as a quadratic function (which is what Newton's Method does). Because the step length in the search direction is effectively fixed for Newton's Method, this can result in indefinite oscillation about the optimum rather than convergence onto it.

A common modification to Newton's Method to overcome this problem (the Newton-Raphson Method) is to use the search direction given by Newton's Method:

$$d_k = -\underline{\underline{H}}^{-1} \nabla f(\underline{x}_k)$$

but to calculate the step length α_k that minimises the Taylor series approximation of the objective function in this search direction, as in, for instance, the Conjugate Gradient Method:

$$\alpha_k = -\frac{d_k^T \nabla f(\underline{x}_k)}{d_k^T \underline{\underline{H}}(\underline{x}_k)\, d_k}$$

and then $\underline{x}_{k+1} = \underline{x}_k + \alpha_k d_k$

4 (a) Minimise $f = 400\left(\frac{1}{2}\pi D^2 + \pi D H\right)$

$\underset{\text{2 ends}}{\frac{1}{2}\pi D^2}$, $\underset{\text{side}}{\pi D H}$

subject to $\frac{\pi D^2 H}{4} - 108\pi = 0$

$$H + \frac{D}{2} - 12 \leq 0$$

(b) $L = 400\left(\frac{1}{2}\pi D^2 + \pi D H\right) + \lambda\left(\frac{\pi}{4}D^2 H - 108\pi\right) + \mu\left(H + \frac{D}{2} - 12\right)$

$$\frac{\partial L}{\partial D} = 400(\pi D + \pi H) + \lambda\left(\frac{1}{2}\pi D H\right) + \frac{1}{2}\mu = 0 \qquad (1)$$

$$\frac{\partial L}{\partial H} = 400(\pi D) + \lambda\left(\frac{\pi D^2}{4}\right) + \mu = 0 \qquad (2)$$

$$\frac{\pi}{4}D^2 H - 108\pi = 0 \qquad (3)$$

$$\mu\left(H + \frac{D}{2} - 12\right) = 0 \qquad (4)$$

(c) If the inequality constraint is active $\mu \neq 0$

If $D = 7.164$ m, (4) $\Rightarrow$ $H = 8.418$ m

$\therefore$ (1): $400\pi(15.582) + \lambda\pi(30.153) + \frac{1}{2}\mu = 0$ (5)

(2): $400\pi(7.164) + \lambda\pi(12.831) + \mu = 0$ (6)

$(5) - \frac{30.153}{12.831}(6)$: $400\pi(-1.253) + \mu(-1.850) = 0$

$\Rightarrow$ $\underline{\underline{\mu = -851.1}}$ as μ is not > 0 this is not a minimum

If $D = 22.256$ m, (4) $\Rightarrow$ $H = 0.872$ m

$\therefore$ (1): $400\pi(23.128) + \lambda\pi(9.704) + \frac{1}{2}\mu = 0$ (7)

(2): $400\pi(22.256) + \lambda\pi(123.83) + \mu = 0$ (8)

$(7) - \frac{9.704}{123.83}(8)$: $400\pi(21.384) + \mu(0.422) = 0$

$\Rightarrow$ $\underline{\underline{\mu = -63.68 \times 10^3}}$ as μ is not > 0 this is not a min.

Thus, neither of the two situations for which the inequality constraint is active given a minimum.

4 (d) If inequality constraint is inactive $\mu = 0$

(2) $\Rightarrow$ $400\pi D = -\lambda \frac{\pi}{4} D^2$

$\therefore$ $D = 0$ (which violates (3))

or $\lambda D = -1600$

(1) $400\pi D + 400\pi H + \frac{1}{2}\lambda \pi D H = 0$

$\therefore$ $400\pi D + 400\pi H - 800\pi H = 0$

$\therefore$ $\underline{\underline{D = H}}$

$\therefore$ (3) $\Rightarrow$ $\frac{\pi}{4} D^3 = 108\pi$ $\Rightarrow$ $\underline{\underline{D = 7.560 \text{ m}}}$

Need to check second-order conditions:

$$\frac{\partial^2 L}{\partial D^2} = 400\pi + \frac{1}{2}\lambda \pi H$$

$$\frac{\partial^2 L}{\partial H^2} = 0 \qquad \frac{\partial^2 L}{\partial D \partial H} = 400\pi + \frac{1}{2}\lambda \pi D$$

Hence for $D = H = 7.560$ m and $\lambda D = -1600$

$$\underline{\underline{H}} = -400\pi \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \underline{\underline{H}} = \text{Hessian of } L$$

Need to find feasible directions at candidate optimum

From (3) $H = \frac{432}{D^2}$ $\therefore$ $\frac{dH}{dD} = -\frac{864}{D^3}$

At $D = H = 7.560$, $\frac{dH}{dD} = -2$

Hence feasible direction at candidate optimum $= \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \underline{d}$

For minimum $\underline{d}^T \underline{\underline{H}}\, \underline{d} > 0$

$$\underline{d}^T \underline{\underline{H}}\, \underline{d} = -400\pi \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= -400\pi \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \underline{\underline{1200\pi}} > 0$$

4 (d) Hence this solution is a minimum.

(e) The Lagrange multiplier λ gives the sensitivity of the optimum to the constraint value, in this case the required volume.

$$\nabla f = |\lambda| \nabla V$$

From (d) $|\lambda| = \frac{1600}{D} = \frac{1600}{7.560} = 211.64$

$$\therefore \nabla f = 211.64 \times 2\pi$$

$$= \text{£}\,1330$$