ENGINEERING TRIPOS PART IIB
ELECTRICAL AND INFORMATION SCIENCES TRIPOS PART II

Saturday 26 April 2003 9 to 10.30

Module 4A12

TURBULENCE

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

(TURN OVER

1 Consider a circular pipe with a right angle bend as shown in Fig. 1. The pipe diameter is $D = 0.01$ m, the mean flow velocity is $\bar{v} = 0.1\ \mathrm{m\,s^{-1}}$ and the fluid (water) has a kinematic viscosity $\nu = 10^{-6}\ \mathrm{m^2\ s^{-1}}$. Assume that there is a fully-developed laminar flow in the upstream part of the pipe, the velocity is purely axial in section A–A′ and its profile is the Poiseuille parabolic function of the radius r

$$u_y = 8\frac{\bar{v}}{D^2}\left(\frac{D^2}{4} - r^2\right).$$

(a) Determine the vorticity distribution in the same section. [30%]

(b) In the part of the pipe downstream of the bend, it is observed that, in addition to the axial velocity component, there is a recirculating flow within cross-section B–B′, as shown in Fig. 1. Give a qualitative explanation of this type of recirculation:

(i) in terms of inviscid vortex dynamics as fluid particles pass the bend; [25%]

(ii) in terms of an 'Ekman pumping' effect. [25%]

(c) Estimate the length downstream of the bend where the recirculation has become negligible. [20%]

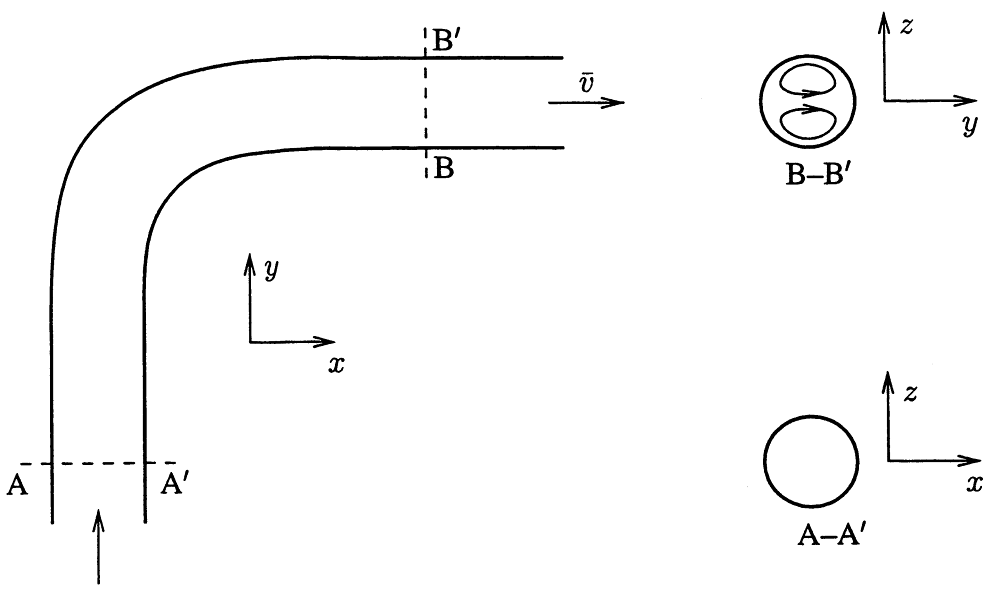

Fig. 1

2 At time zero, a three-dimensional incompressible flow is such that its vorticity has a z-component only:

$$\omega_z = A_0 \text{ Re}\left[e^{i(k_0 x + k_0 y)}\right],$$

and its velocity components are

$$\begin{aligned} u_x &= -\alpha x + v_x, \\ u_y &= v_y, \\ u_z &= \alpha z, \end{aligned}$$

where v_x and v_y depend on x and y only and have a zero space-average.

(a) Show that the vorticity is entirely due to v_x and v_y and that (v_x, v_y) must form a divergenceless vector field. [20%]

(b) Find expressions for v_x and v_y at time zero, by assuming that they are of the form $e^{i(k_0 x + k_0 y)}$. [30%]

(c) Sketch this flow, showing the vortex tubes. [20%]

(d) The subsequent evolution of the flow is supposed to obey the inviscid fluid equation, the z-component of the vorticity being of the following form:

$$\omega_z = A(t) \text{ Re}\left[e^{i(k_x(t) x + k_0 y)}\right].$$

Determine the functions $A(t)$ and $k_x(t)$ and give a physical interpretation of the flow evolution. [30%]

(TURN OVER

3 Consider a two-dimensional turbulent flow in a circular domain $\mathcal{S}$ with no forcing. The fluid is viscous (although its Reynolds number is large) and, at the circular boundary $\mathcal{C}$, a condition of zero shear stress is imposed as shown in Fig. 2.

(a) Give a qualitative argument indicating that the vorticity flux $\Phi = \int_{\mathcal{S}} \omega \, dS$ is strictly constant in time. [20%]

(b) Obtain an expression for the decay of the total kinetic energy $E = \int_{\mathcal{S}} u^2/2 \, dS$ and show that this decay is proportional to viscosity for a given initial velocity field. [20%]

(c) Show that the enstrophy $D = \int_{\mathcal{S}} \omega^2 \, dS$ decreases and that this decrease can potentially take place on a short time scale of the order of the flow turnover time. [20%]

(d) For a period of time after the flow is initially generated, large compared to the turnover time and small compared to the viscous timescale of decay, it can be assumed that the vorticity flux Φ and kinetic energy E have remained constant and that the enstrophy D has reached the minimum value possible. Now consider axisymmetric flows only. Show that the condition on the initial energy E_0 and vorticity flux Φ_0, such that the flow of minimum enstrophy is solid body rotation, is $\Phi_0^2 = 16\pi E_0$. [20%]

(e) Sketch the axisymmetric vorticity distribution expected after the same time delay as considered in (d), when $\Phi_0^2 > 16\pi E_0$ and when $\Phi_0^2 < 16\pi E_0$. [20%]

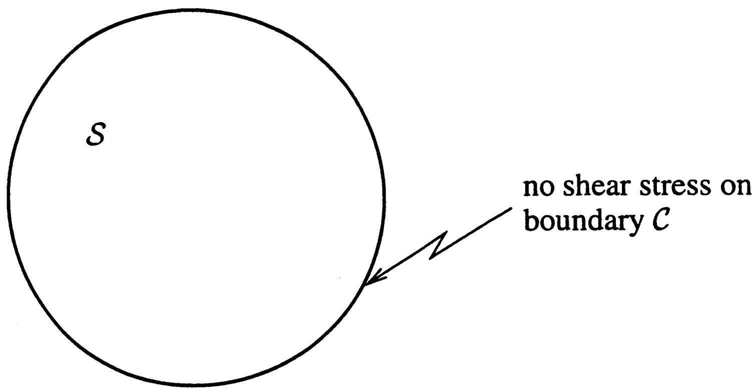

Fig. 2

4 When a $k - \epsilon$ model is used the turbulent specific kinetic energy k and the specific energy dissipation rate ϵ are computed at each computational grid point in addition to the mean velocity field. The governing equations for k and ϵ are solved subject to the appropriate boundary conditions. Consider the boundary conditions for k, ϵ and the tangential velocity next to a smooth wall, and assume that the closest grid points lie within the logarithmic region.

(a) Show that k is proportional to the square of the friction velocity $u_* = \sqrt{\tau_w/\rho}$, where τ_w is the shear stress at the wall. [20%]

(b) Show how the tangential mean velocity U is related to the friction velocity and the distance from the wall. Hence, using (a), obtain a relationship between k and the mean velocity U at the first grid point, situated at a distance y_0 from the wall. [20%]

(c) Using the local balance between energy production and dissipation in the logarithmic region, obtain an expression for ϵ at the first grid point. [30%]

(d) A third relationship involves the derivative of the mean velocity profile with respect from the distance to the wall. What is this relationship? [30%]

END OF PAPER