ENGINEERING TRIPOS PART IIB
ELECTRICAL AND INFORMATION SCIENCES TRIPOS PART II

Monday 28 April 2003 9 to 10.30

Paper 4C7

RANDOM AND NON-LINEAR VIBRATIONS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

Candidates may bring their notebooks to the examination.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

(TURN OVER

1 (a) A Gaussian random process $x(t)$ has autocorrelation function $R_{xx}(\tau)$ and double-sided spectrum $S_{xx}(\omega)$. A second random process is defined as $z(t) = x^2(t)$. Write down the definition of the autocorrelation function of $z(t)$ and by using the first formula given at the end of the question express this result in terms of $R_{xx}(\tau)$. [20%]

(b) By expressing $R_{xx}(\tau)$ as the Fourier Transform of $S_{xx}(\omega)$, and by making use of the second formulae given at the end of the question, show that

$$S_{zz}(\omega) = R_{xx}^2(0)\delta(\omega) + 2\int_{-\infty}^{\infty} S_{xx}(\omega')S_{xx}(\omega'+\omega)\,\mathrm{d}\omega'.$$ [35%]

(c) A floating offshore structure is subjected to an incident wave spectrum of the form

$$\begin{aligned} S_{xx}(\omega) &= A, \quad && 0.2 \le |\omega| \le 0.3 \text{ rad s}^{-1}, \\ &= 0, && \text{elsewhere.} \end{aligned}$$

It is found that the wave force has a non-linear component given by $z(t) = x^2(t)$. Derive an expression for the double-sided spectrum of $z(t)$, and sketch your result. [35%]

(d) The floating structure has a natural frequency of 0.02 rad s^{-1}. Explain why a large response may be observed, even though the natural frequency lies outside the frequency range of the wave spectrum. [10%]

Useful formulae:

(i) If x_1, x_2, x_3, and x_4 are Gaussian random variables, then

$$\mathrm{E}[x_1x_2x_3x_4] = \mathrm{E}[x_1x_2]\mathrm{E}[x_3x_4] + \mathrm{E}[x_1x_3]\mathrm{E}[x_2x_4] + \mathrm{E}[x_1x_4]\mathrm{E}[x_2x_3].$$

(ii) One expression for the delta function is

$$\delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau}\,\mathrm{d}\tau.$$

2 (a) Two uncoupled oscillators are shown in Fig. 1. Each oscillator has mass M, stiffness K, and damping constant C, and the initial clearance between the oscillators is d. The oscillators are subject to forces $F(t)\pm\alpha G(t)$ as shown in the Figure, where $F(t)$ and $G(t)$ are *statistically independent* white noise random processes, each of spectrum S_0. Write down the equation of motion of each oscillator, and hence find the mean squared values of the displacements x_1 and x_2. [25%]

(b) Derive expressions for the variance of the relative displacement and the variance of the relative velocity between the oscillators. [20%]

(c) For the case $M = 1$ kg, $K = 4$ N m^{-1}, and $C = 0.04$ N s m^{-1}, $\alpha = 0.5$, $d = 0.1$ m and $S_0 = 3\times10^{-5}$ N^2 rad^{-1} s find the probability that the oscillators will impact at least once in a 1 hr period. [30%]

(d) The force on the left hand oscillator is now modified to $\gamma(F+\alpha G)$ where γ is a constant. Show that the variance of the relative displacement is minimised when $\gamma = (1-\alpha^2)/(1+\alpha^2)$. [25%]

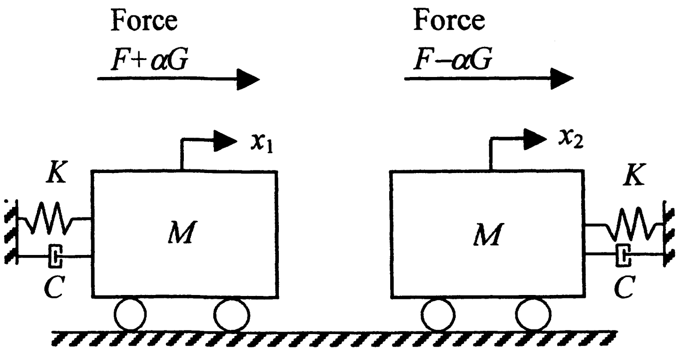

Fig. 1

3 A particle of unit mass moves along the x-axis under the influence of a potential

$$U(x) = x^4 - 2x^2 + kx$$

where k is a constant.

(a) For the case $k=0$ find the singular points of this system. By linearising the governing equation in the vicinity of each singular point, establish the nature of the singular points. Hence sketch the phase portrait of the system. Indicate clearly any separatrices, and describe briefly the behaviour of the system in each region of the phase plane. [40%]

(b) When $k \neq 0$, show by considering the graph of the function $f(x) = 4x^3 - 4x$ that there is a value k_{crit} of k at which the number of critical points changes. Find this value. [20%]

(c) Without detailed calculations, explain how the phase portrait will change as k varies. Sketch examples for (i) $0 < k < k_{crit}$ and (ii) $k > k_{crit}$. [40%]

4 A simple pendulum consists of a small mass m hanging beneath a table on the end of a length of string. The string passes through a small hole in the table, to a machine on top of the table which causes the length L of the string to be periodically varied so that

$$L = L_0 + a\sin\omega t$$

where L_0 and a are constants.

(a) If the pendulum swings in one plane only and the angle of the pendulum to the downward vertical is θ, show that the free motion of the pendulum is governed by the equation

$$(L_0 + a\sin\omega t)\ddot{\theta} + 2a\omega\dot{\theta}\cos\omega t + g\sin\theta = 0$$

where g is the acceleration due to gravity. [25%]

(b) Under the assumptions

$$|\theta| << 1 \quad \text{and} \quad a << L_0$$

use the method of iteration to obtain an approximate solution to this governing equation correct to first order in a/L_0. [50%]

(c) Under what conditions does the pendulum exhibit unstable growing oscillations? Does the phase of the pendulum oscillation relative to that of the length variation influence the rate of unstable growth? [25%]

END OF PAPER