ENGINEERING TRIPOS PART IIB
ELECTRICAL AND INFORMATION SCIENCES TRIPOS PART II

Saturday 26 April 2003 9 to 10.30

Module 4C12

WAVE PROPAGATION

Answer not more than ***two*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Candidates may bring their notebooks to the examination.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

1 (a) A one-dimensional wave-bearing system has a dispersion relation

$$\omega^2 = Ak^\lambda$$

where ω is frequency, k is wavenumber, and A and λ are constants. Calculate the phase velocity and group velocity of these waves, and the ratio between the two. Name familiar systems which illustrate the cases (i) $\lambda < 2$; (ii) $\lambda = 2$; and (iii) $\lambda > 2$. [20%]

(b) Figure 1 shows a snapshot of the wave pattern found in a system like that of part (a) at a particular instant. It consists of a carrier wave of constant wavelength, modulated by a Gaussian envelope shape. The waves are travelling from left to right, and the dispersion law has $\lambda = 4$. Copy the figure, and sketch below it a snapshot of the wave pattern a short time later, labelled to show the effects of phase velocity and group velocity. [20%]

(c) Bending waves in a two-dimensional thin sheet of laminate material obey the differential equation

$$-\frac{\partial^2 w}{\partial t^2} = D_1 \frac{\partial^4 w}{\partial x^4} + D_2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_3 \frac{\partial^4 w}{\partial y^4}$$

where $w(x, y, t)$ is transverse displacement and D_1, D_2 and D_3 are constants. A sinusoidal plane wave propagating in a certain direction can be written in the form

$$w = e^{i(k_1 x + k_2 y - \omega t)}.$$

What is the angle between the wave crests and the x axis? What is the phase velocity vector $\mathbf{c}_p$? [20%]

(d) The group velocity vector of this system is given by

$$\mathbf{c}_g = \left[\frac{\partial \omega}{\partial k_1}, \frac{\partial \omega}{\partial k_2} \right].$$

Show that the group velocity is not in general in the same direction as the phase velocity. [20%]

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(e) Is it possible to find positive values of D_1, D_2 and D_3 such that the group velocity is perpendicular to the phase velocity for a particular angle of propagation? [20%]

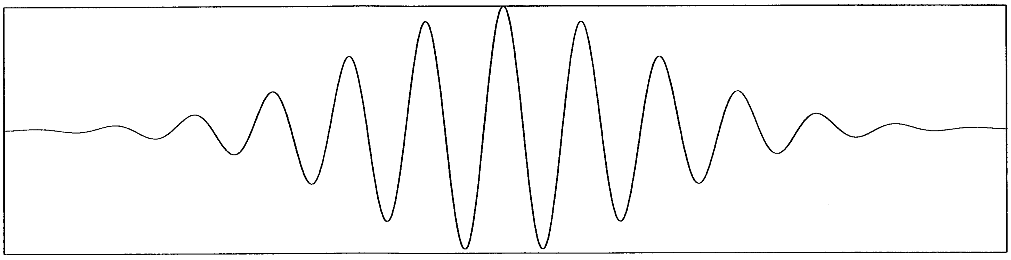

Fig. 1

2 A slender uniform bar that is free to move axially is initially stationary before one end is struck in an axial direction by the end of a second slender bar. The second bar is moving in the coaxial direction with initial speed V_0. The two bars are each of length L but the cross-sectional areas A_i and acoustic impedances $\rho_i c_i$, $i = 1, 2$ are different as shown in Fig. 2. Wave speed $c_2 \geq c_1$ and impact occurs at time $t = 0$.

(a) Write expressions that relate stresses and particle velocities across the interface where impact occurs ($x = 0$). [20%]

(b) Find the stress distribution in both bars for times $t < L / c_2$. Express the stress as a function of the parameter ratio $\beta = \rho_2 c_2 A_2 / \rho_1 c_1 A_1$. Sketch the stress distribution at time $t = \gamma L / c_2$, $\gamma < 1$ and on the sketch indicate the stress magnitude at wavefronts. For your sketch assume $\beta > 1$. [30%]

(c) Write an expression for the momentum of the two bars at time $t = \gamma L / c_2$, $\gamma < 1$ and show that this equals the initial momentum. [20%]

(d) Write an expression for the total mechanical energy of this system at time $t = \gamma L / c_2$, $\gamma < 1$ and show that this equals the initial kinetic energy. [30%]

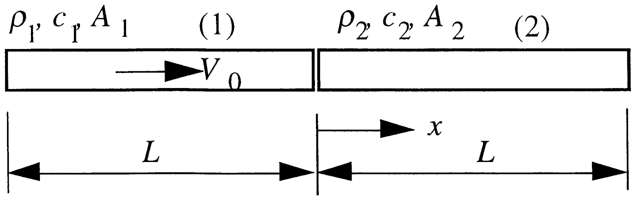

Fig. 2

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3 Figure 3 shows a plane compressive wave in material 1 that is incident on the interface between two elastic solids. The incoming wave is travelling in direction $\mathbf{e}_{z'}$; i.e. at angle of incidence θ_1 from the normal to the interface. Both materials 1 and 2 have negligible shear stiffness; hence each material satisfies a constitutive relation of the form $\sigma_m = Ke$ where σ_m is the mean stress and e is the dilatation. The incoming wave has particle displacement $u_{z'}$ solely in the direction of wave propagation.

(a) Derive a non-dispersive wave equation (equation of motion) for pressure waves through these media and hence obtain an expression for the wave speed, c. [20%]

(b) Let the surface $z = 0$ be an interface between materials with equal compressibility K but distinct densities $\rho_1 = \rho$ and $\rho_2 = \rho / 3$. The incident plane wave has wave vector $k_1\mathbf{e}_{z'}$. Find the direction θ_2 of the transmitted wave as a function of the angle of incidence θ_1. Also find the ratio of wavenumbers k_2 / k_1 for transmitted and incident waves. [20%]

(c) Express the boundary conditions at the interface in terms of components of incident and transmitted waves. [20%]

(d) For amplitude A_1 of the incident wave, find the amplitudes A_2 of the reflected wave and C_2 of the transmitted wave as functions of the angle θ_1. [25%]

(e) Express an upper bound on the range of incident conditions where this solution is valid and explain the cause of this upper bound. [15%]

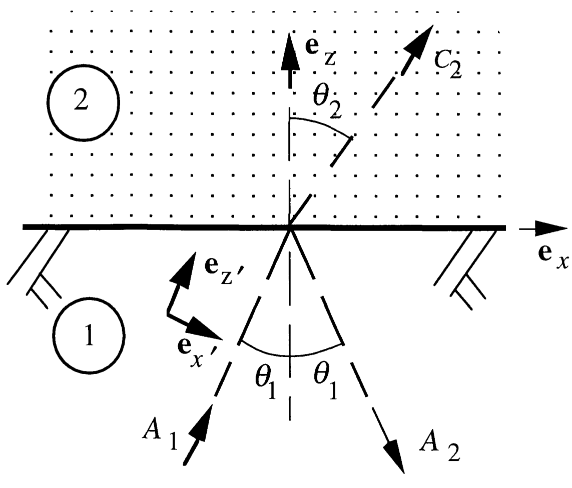

Fig. 3

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