ENGINEERING TRIPOS PART IIB
ELECTRICAL AND INFORMATION SCIENCES TRIPOS PART II

Saturday 26th April 2003 2.30-4.00

Module 4F2

ROBUST MULTIVARIABLE CONTROL

Answer not more than ***two*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

(TURN OVER

1 (a) State the Small Gain Theorem, defining any terms used. [20%]

(b) Show, with the aid of a diagram, that a controller K stabilizes all systems of the form

$$G(I + W\Delta)^{-1}, \ \|\Delta\|_\infty < \epsilon$$

if, and only if,

$$\|(I + KG)^{-1}W\|_\infty \leq 1/\epsilon.$$

W and Δ are taken to be stable transfer function matrices, and a standard negative feedback configuration is assumed. [40%]

(c) Consider the uncertain second order system

$$G_1 = \frac{s}{s^2 + (0.1 + \delta_1)s + 1 + \delta_2},$$

where δ_1 and δ_2 represent uncertainty in the damping and stiffness respectively and are only known to satisfy

$$|\delta_1| < 1, \ |\delta_2| < 1.$$

Using the result in part (b), show that the constant controller $K = 1.9$ is guaranteed to stabilize G_1.

Find directly the precise range of values of δ_1 and δ_2 for which K stabilizes G, and describe how the Small Gain Theorem may be extended to provide a less conservative result for this example. [40%]

2 The value function, $V(x,t)$, for a differential game

$$\dot{x} = f(x,u,w), \quad x(0) = x_0$$

with $x \in \mathbb{R}^n$, $w \in W \subseteq \mathbb{R}^p$, $u \in U \subseteq \mathbb{R}^m$ and cost

$$J(x_0, w(\cdot), u(\cdot)) = \int_0^T c(x(t), w(t), u(t))dt + J_T(x(T))$$

can be computed by solving Isaacs equation

$$\frac{\partial V}{\partial t}(x,t) + \max_{w \in W} \min_{u \in U} \left(c(x,w,u) + \frac{\partial V}{\partial x}(x,t) f(x,w,u) \right) = 0$$

with boundary condition $V(x,T) = J_T(x)$.

(a) Assume $x \in \mathbb{R}$, $U = \mathbb{R}$, $W = \mathbb{R}$, $T = 1$,

$$\begin{aligned} f(x,w,u) &= x + w + u \\ c(x,w,u) &= u^2 - w^2 \\ J_1(x) &= x^2. \end{aligned}$$

(i) Assume $V(x,t) = p(t)x^2$ for some $p(t)$ to be determined. Write Isaacs equation for this system [10%]

(ii) By taking derivatives with respect to u and w eliminate the min and the max from Isaacs equation. Hence derive a Riccati differential equation for $p(t)$. [20%]

(iii) Compute the solution of the Riccati differential equation. Hence derive an explicit formula for $V(x,t)$. [30%]

(b) Assume now that $x \in \mathbb{R}$, $U = [-1,1]$, $W = [-1,1]$, $T = 1$ and

$$\begin{aligned} f(x,w,u) &= w + (|x| + 1)u \\ c(x,w,u) &= 0 \\ J_1(x) &= 1 - x^2. \end{aligned}$$

(i) Write Isaacs equation for this system. [10%]

(ii) Show that $V(x,t) = 1 - x^2 e^{-2(t-1)}$ is a solution to Isaacs equation. (*Hint*: Distinguish three cases: $x > 0$, $x = 0$ and $x < 0$.) [30%]

3 Consider the feedback arrangement of Fig. 1.

(a) Assume that

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \begin{bmatrix} w(s) \\ u(s) \end{bmatrix}.$$

Derive the lower fractional transformation $\mathcal{F}_l(P(s), K(s))$ relating $z(s)$ to $w(s)$ for this system. [20%]

(b) Assume now that the plant $P(s)$ is in the standard state space form

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = Ax + B_1 w + B_2 u$$

$$z = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = C_1 + D_{12} u$$

$$y = x.$$

(i) Write the Control Algebraic Riccati Equation (CARE) for this system and compute all its real, symmetric solutions.
(*Hint*: There may be multiple solutions to this equation. Recall that the CARE is $XA + A^T X + C_1^T C_1 - X B_2 B_2^T X = 0$.) [20%]

(ii) Write the expression for the state feedback controller corresponding to each solution of (b)(i). [10%]

(iii) For each of the controllers of (b)(ii) write the closed loop system in state space form. Hence determine which controllers are stabilizing. [20%]

(iv) For the stabilizing controllers of (b)(iii) compute $\mathcal{F}_l(P(s), K(s))$.
(*Hint*: You can assume that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc \neq 0$ then $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.) [20%]

(v) Compute

$$\min_{K(s) \text{ stabilizing}} \|\mathcal{F}_l(P(s), K(s))\|_2 .$$

[10%]

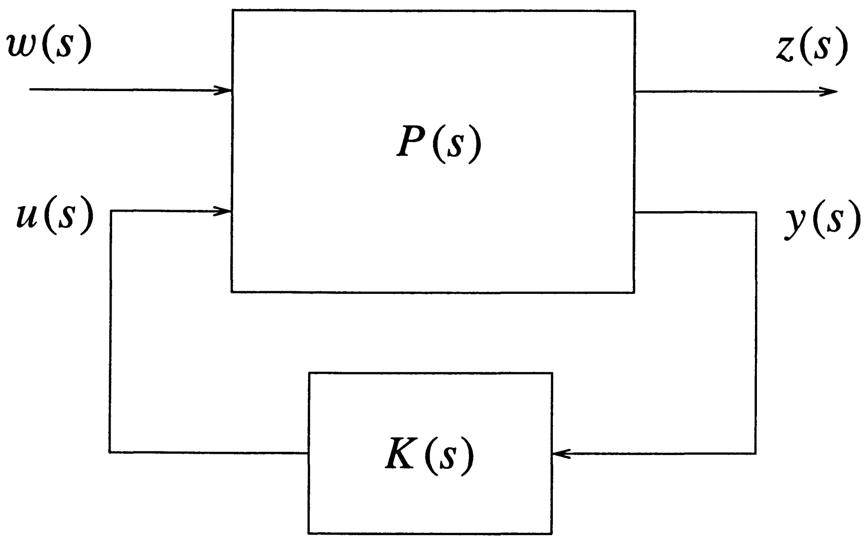

Fig. 1

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