ENGINEERING TRIPOS PART IIB
ELECTRICAL AND INFORMATION SCIENCES TRIPOS PART II

Tuesday 22 April 2003 9 to 10.30

Module 4F3

NON-LINEAR AND HYBRID SYSTEMS

Answer not more than **three** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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1 (a) Show that the describing function of the saturation nonlinearity

$$\text{sat}(e) = \begin{cases} -1 & \text{if } e < -1 \\ e & \text{if } |e| \leq 1 \\ +1 & \text{if } e > +1 \end{cases}$$

is

$$N(E) = \begin{cases} 1 & \text{if } E \leq 1 \\ \frac{2}{\pi}\left[\sin^{-1}\left(\frac{1}{E}\right) + \frac{1}{E}\sqrt{1 - \frac{1}{E^2}}\right] & \text{if } E > 1. \end{cases}$$

Give an intuitive argument for the fact that $N(E) \leq 1$. [35%]

(b) Consider the feedback system shown in Figure 1, where the saturation nonlinearity is the same as the one defined in part (a). Show that describing function analysis predicts a limit cycle oscillation if $k > 2$ and find the frequency of this oscillation. [30%]

(c) Use the circle criterion to prove that the feedback system shown in Figure 1 is globally asymptotically stable if $k < 1/2$. [35%]

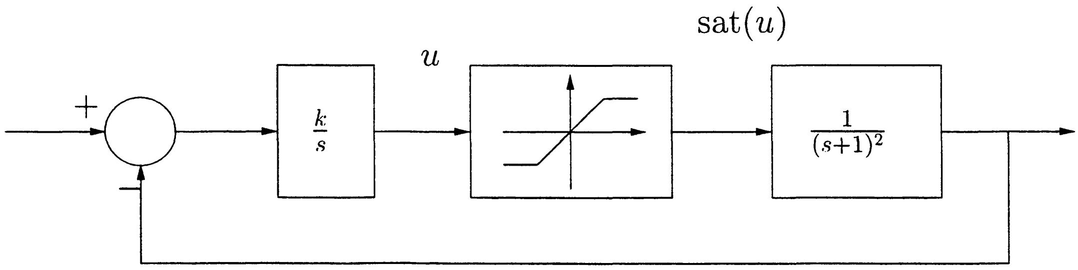

Fig. 1

2 The controlled van der Pol oscillator is given by the equations

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + (1 - x_1^2)x_2 + (1 + x_1^2 + x_2^2)u$$

where as usual $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$ are the states of the system and $u \in \mathbb{R}$ is a control.

(a) What is the dimension of this system? [5%]

(b) Assume that $u = 0$. Determine the equilibria of the system and investigate their stability using linearisation. [25%]

(c) Assume now that the control u is determined by linear state feedback of the form

$$u = ax_1 + bx_2,$$

where a and b are constant gains.

(i) Determine all the equilibria of the closed loop system. [25%]

(ii) Assume that $a = 0$. Using linearisation choose a value for b such that the closed loop system has a locally asymptotically stable equilibrium at $x_1 = x_2 = 0$. [25%]

(d) Your friend claims that she can globally asymptotically stabilize the equilibrium $(0, 0)$ using the controller

$$u = \frac{x_1^2 x_2 - 3x_2}{1 + x_1^2 + x_2^2}$$

Do you believe her? Justify your answer. [20%]

3 (a) Consider a hybrid automaton $H = (Q, X, f, \mathit{Init}, \mathit{Dom}, E, G, R)$. Let (τ, q, x) denote a finite execution of this automaton, with $\tau = \{[\tau_i, \tau_i']\}_{i=0}^N$ a hybrid time set, $q = \{q_i(\cdot)\}_{i=0}^N$ a sequence of functions $q_i(\cdot) : [\tau_i, \tau_i'] \rightarrow Q$ and $x = \{x_i(\cdot)\}_{i=0}^N$ a sequence of functions $x_i(\cdot) : [\tau_i, \tau_i'] \rightarrow X$.

(i) Explain what the statement "the state $(\hat{q}, \hat{x}) \in Q \times X$ is reachable" means. [10%]

(ii) Explain what the statement "the set of states $M \subset Q \times X$ is invariant" means. [10%]

(iii) Consider a set of states $F \subseteq Q \times X$. Assume there exists an invariant set $M \subseteq Q \times X$ such that $\mathit{Init} \subseteq M \subseteq F$. What can we conclude about the reachable states of the system? [10%]

(b) Consider a hybrid automaton $H = (Q, X, f, \mathit{Init}, \mathit{Dom}, E, G, R)$ shown in Figure 2.

(i) What are the sets of discrete states Q and continuous states X? What is the set of initial states $\mathit{Init} \subseteq Q \times X$? [10%]

(ii) Consider $(q_0, x_0) \in \mathit{Init}$ with $q_0 = \mathit{Left}$. Show that if $x(t)$ is the solution of the differential equation

$$\begin{aligned} \dot{x} &= f(q_0, x) \\ x(0) &= x_0 \end{aligned}$$

then $(q_0, x(t)) \in \mathit{Init}$ as long as $x(t) \in \mathit{Dom}(q_0)$.
(*Hint*: Consider the derivative of $\cos(1 + x_1(t)) + \frac{x_2(t)^2}{2}$.) [20%]

(iii) Consider $(q_0, x_0) \in \mathit{Init}$ with $q_0 = \mathit{Left}$. Show that if a discrete transition takes place from (q_0, x_0) the state after the discrete transition will also be in *Init*. [20%]

(iv) Assume that the conclusions of part (b)(ii) and (b)(iii) also hold when $q_0 = \mathit{Right}$. What can you conclude about the set of reachable states of H? You do not need to provide a formal induction argument; an informal discussion will suffice. [20%]

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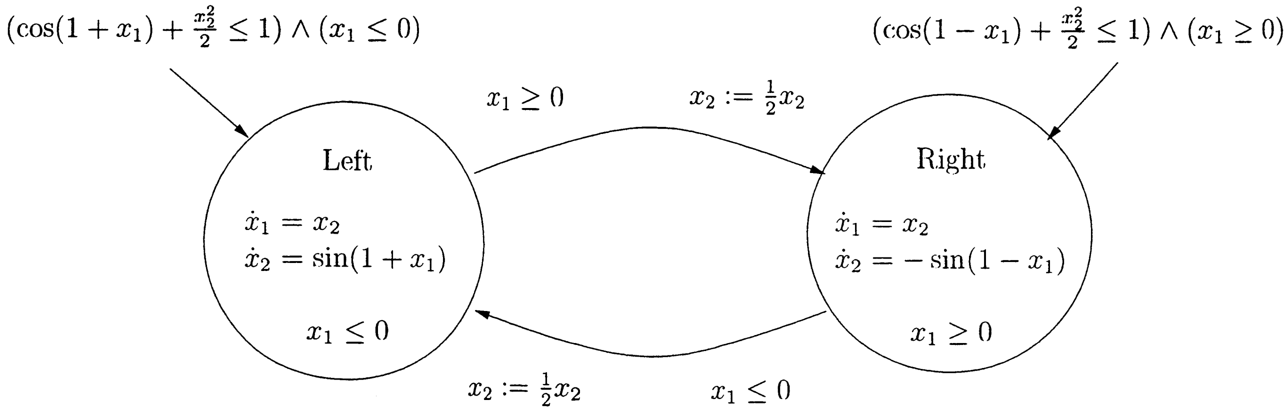

Fig. 2

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4 (a) Briefly explain the meaning of the terms *model checking* and *timed automaton*. What property of timed automata makes them amenable to model checking methods? [30%]

(b) Consider the dynamical system

$$\dot{x} = \mu x - x^3,$$

where $x \in \mathbb{R}$ and μ is a real valued, constant parameter.

(i) What is the dimension of this system? [5%]

(ii) Determine all equilibria of the system. Distinguish two cases:

1. $\mu > 0$.
2. $\mu \leq 0$.

[20%]

(iii) Using linearisation determine the stability of all equilibria when

1. $\mu > 0$.
2. $\mu < 0$.

[25%]

(iv) Draw a plot showing the positions of the equilibria as a function of μ. Be sure to include both positive and negative values of μ.
(*Hint*: if multiple equilibria exist for some values of μ your plot may have several branches.) [20%]

END OF PAPER