ENGINEERING TRIPOS PART IIB
ELECTRICAL AND INFORMATION SCIENCES TRIPOS PART II

Monday 28th April 9 to 10.30

Module 4F7

DIGITAL FILTERS AND SPECTRUM ESTIMATION

Answer not more than **three** *questions.*

All questions carry the same number of marks.

The **approximate** *percentage of marks allocated to each part of a question is indicated in the right margin.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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1 (a) An Nth-order analogue highpass Butterworth filter, with -3dB cutoff frequency ω_c radians/sec has a frequency response $H(j\omega)$ which satisfies

$$|H(j\omega)|^2 = \frac{1}{1+(\omega_c/\omega)^{2N}}$$

It is required to design, using the Bilinear Transform method, a digital highpass filter with a gain of 0dB at one half of the sampling frequency, a -3dB cutoff frequency of 0.75π radians/sample, and an attenuation of at least 30 dB in the stopband, which runs from 0 to 0.25π radians/sample. Show that a second order Butterworth filter ($N = 2$) will satisfy this requirement. [30%]

(b) The transfer function of a second order lowpass Butterworth filter with cutoff frequency 1 radian/sec is

$$H(s) = \frac{1}{1+\sqrt{2}s+s^2}$$

Using the lowpass-to-highpass mapping $s \longrightarrow \omega_c/s$ and the Bilinear Transform, calculate the transfer function of a digital filter to meet the requirement set out in part (a). [40%]

Sketch how this filter would be implemented using a second order (biquadratic) section, showing clearly the coefficient values used. [10%]

(c) State the meaning of 'overflow' and 'overflow oscillation' in the context of IIR digital filters implemented using fixed-point arithmetic.

If only the feedback paths of the biquadratic section in part (b) are implemented (and not the feedforward, that is, numerator coefficients), the frequency response of the resulting filter is found to have a maximum amplitude of 2.5981.

Using this information, and assuming that the same fixed-point arithmetic precision and wordlength is used for all data values in the filter, deduce the scale factors required to apply "frequency response scaling" in order to both reduce the risk of overflow and keep the data values as large as possible to maximise Signal to Noise Ratio. Explain clearly how the scaling is applied.

Is this scaling sufficient to prevent overflow oscillation, and if not, what other steps must be taken? [20%]

2 (a) Briefly describe the steps in the window method of FIR filter design. [10%]

(b) A lowpass FIR filter is required, with a -3dB passband corner frequency no lower than 0.3π radian/sample, and a stopband attenuation of at least 45dB for frequencies above 0.43π radian/sample. The filter is to be designed using the window method, and an odd-length Hamming window is to be used. Tabulated data suggests that the ratio (transition width/sample frequency) will be approximately $2.7/N$, where N is the filter length.

Determine the minimum filter length (that is, window length) that in fact meets the specification, by following the window method. Since you cannot carry out the normal computational steps (which, given an ideal lowpass filter with cutoff frequency ω_c rad/sample, involve computing the filter coefficients using the chosen window length N, and then determining the actual frequencies at which the filter response is -3 dB or -45 dB), use the following empirically determined formulae instead:

- Filter response = -3dB at frequency $\omega_c - \frac{1.2}{N-10}$ radians/sample
- Filter response = -45dB at frequency $\omega_c + \frac{11.5}{N+1}$ radians/sample

[60%]

(c) Consider an LMS filter of length L, with white input signal $x(k)$ of variance σ_X^2. What are the limits on the step size μ for convergence to occur, and what is a suitable value μ_0 to give fast convergence?

Describe briefly how the value of μ affects all aspects of the behaviour of the filter, and state what happens if the chosen value of μ is (i) slightly higher, (ii) slightly lower, (iii) much higher or (iv) much lower than μ_0.

What are the consequences if the input signal $x(k)$, still of variance σ_X^2, is coloured?

What adaptive filter algorithms are there which perform better than LMS when the input signal is coloured, and what advantages and disadvantages do they have? [30%]

3 (a) Describe the *periodogram* method for power spectrum estimation of a stationary random process. Your description should include a discussion of its advantages and weaknesses. Briefly describe, without detailed mathematics, three methods for improving the performance of the periodogram estimate. [40%]

(b) A power spectrum estimate for a stationary ergodic random process is to be made using a *modified* autocorrelation function estimate of the form:

$$\hat{R}_{XX}[k] = \frac{1}{N}\sum_{n=0}^{N-1-k}(w_n x_n)(w_{n+k}x_{n+k}), \text{ for } k = 0,1,\ldots,N-1$$

and

$$\hat{R}_{XX}[k] = \hat{R}_{XX}[-k], \text{ for } k = -1,-2,\ldots,-N+1$$

where the x_n are measured values drawn from the random process and w_n is some suitably chosen windowing function satisfying $\sum_{n=0}^{N-1} w_n^2 = N$, and $w_n = 0$ for $n < 0$ and $n > N-1$.

Determine whether this modified autocorrelation function estimate is unbiased. [15%]

Show that the expected value of the corresponding power spectrum estimate is:

$$E[\hat{S}_X(e^{j\omega T})] = \frac{1}{2\pi N} S_X(e^{j\omega T}) * |W(e^{j\omega T})|^2$$

where $S_X(e^{j\omega T})$ is the true power spectrum of the random process, $W(e^{j\omega T})$ is the DTFT of the window function w_n, and $*$ denotes convolution in frequency. [30%]

Comment on the effect of window function choice and explain how frequency resolution of the method varies with N. [15%]

4 (a) Describe briefly the *parametric* approach to power spectrum estimation. You should include in your description the ARMA model, the AR model, and the corresponding power spectra for these models. What advantages and disadvantages does the parametric method have when compared with non-parametric approaches such as the periodogram? [40%]

(b) It is required to estimate the parameters of an AR model from a finite length of data. The least squares criterion is adopted in which the prediction error squared e_n^2 is minimised over time points n_i through n_f :

$$E = \sum_{n=n_i}^{n_f} e_n^2$$

Show that the optimal estimate of the parameter vector $\mathbf{a}$ under the least squares criterion can be expressed as:

$$\mathbf{a} = -(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{x}$$

where $\mathbf{x}$ and $\mathbf{X}$, which are a vector and matrix containing observed data values, should be carefully defined.

State how the *covariance* and *autocorrelation* methods may be obtained from this equation by appropriate choice of the initial and final conditions. What are the properties of the two methods? [30%]

(c) The first N values of a signal x_n are given by:

$$1, \beta, \beta^2, \ldots, \beta^{N-1}$$

Estimate a first order (i.e. having one single coefficient a_1) AR model from this data using the covariance method. Determine the prediction error at this estimated value and comment on this result. Does the estimated model always correspond to a stable all-pole filter? [30%]

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