ENGINEERING TRIPOS PART IIB
ELECTRICAL AND INFORMATION SCIENCES TRIPOS PART II

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Wednesday 30 April 2003 2.30 to 4.00

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Module 4M12

COMPLEX ANALYSIS AND VARIATIONAL METHODS

*Answer not more than **three** questions.*

*All questions carry the same number of marks.*

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

*Candidates may bring their notebooks to the examination.*

**You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator**

(TURN OVER

1 (a) Show that the complex function $w = a \sin z$ can be inverted to give

$$iz = \ln\left(\frac{iw}{a} + \left[1 - \left(\frac{w}{a}\right)^2\right]^{1/2}\right)$$

[20%]

(b) Sketch the image in the $w$-plane of the lines OA, AB, CD, OE, FG, HI, OF and OJ in the $z$-plane, as shown in Figure 1. [40%]

(c) The domain BGID in the $z$-plane can be extended to an infinite strip of width $\pi$ along the positive and negative Im($z$) axis. The boundaries of this extended domain map into two rigid walls placed along the axis Im($w$)=0; one wall is located at $a < \mathrm{Re}(w) < \infty$, and the other is at $-\infty < \mathrm{Re}(w) < -a$, with a gap between the walls over ($-a < \mathrm{Re}(w) < a$, Im($w$)=0). Consider inviscid, incompressible fluid flow in the direction of positive Im($w$) through this gap; the flow velocity approaches zero far from the gap (as $|w| \to \infty$). The total flow rate through the gap is given by $Q$ (per unit depth into the page). Obtain an expression for the complex potential $\Omega(w) = \phi + i\psi$ for the flow. [40%]

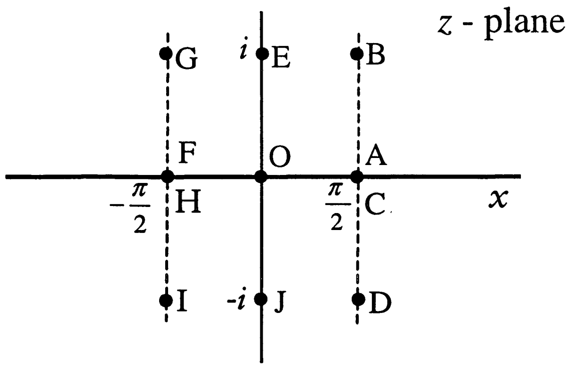


Figure 1

2 Calculate the following integrals using contour integration and the Residue Theorem:

(a) $$\int_0^\infty \frac{x^2}{\left(x^2+4\right)^2}dx$$ [50%]

(b) $$PV\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2-9}dx\ , \qquad a \text{ real}$$ [50%]

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3 (a) The displacement of a two-degree-of-freedom conservative dynamic system is described by two generalised coordinates $q_1(t)$ and $q_2(t)$. Hamilton's principle states that the motion of the system over any time interval $t_0 \leq t \leq t_1$ must satisfy the variational condition

$$\delta\int_{t_0}^{t_1}\{T(\dot{q}_1,\dot{q}_2,q_1,q_2)-V(q_1,q_2)\}\,\mathrm{d}t=0,$$

where $T$ is the kinetic energy and $V$ is the potential energy. Show that the equations of motion of the system have the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial V}{\partial q_i}=0, \qquad i=1,2.$$

[30%]

(b) The system described above is now subjected to a constraint that enforces the condition

$$C(q_1,q_2)=0,$$

where $C$ is some continuous function of $q_1(t)$ and $q_2(t)$. Show how Hamilton's principle can be modified to allow for this constraint, and hence derive the equations of motion that govern the constrained system. [35%]

(c) The planar motion a particle of mass $m$ is described by polar coordinates $r(t)$ and $\theta(t)$. The potential energy of the particle is given by

$$V=\frac{1}{2}k(r\cos\theta)^2,$$

where $k$ is a constant. The motion of the particle is subject to the constraint

$$r\sin\theta=a,$$

where $a$ is a constant. Use your result to part (b) to derive the equations of motion of the particle. Hence show that the system is dynamically equivalent to a simple mass $m$ on a spring of stiffness $k$. [35%]

4 (a) A vibrating balloon can be considered to be an acoustic volume $V$ surrounded by a surface $S$ that consists of a tensioned membrane. For harmonic vibrations of frequency $\omega$, the pressure $p$ in the balloon and the displacement $u$ of the membrane (in the direction of the normal $\mathbf{n}$ to the surface $S$) satisfy the variational statement

$$\delta L = 0 .$$

The functional $L$ is given by

$$L = \frac{1}{2\rho_0 c^2}\int_V \left[p^2 - (c/\omega)^2 \nabla p.\nabla p\right] \mathrm{d}V + \frac{1}{2}\int_S \left[T\nabla u.\nabla u - \omega^2 m u^2 - 2pu\right] \mathrm{d}S ,$$

where $T$ is the tension in the membrane, $m$ is the membrane mass per unit area, $c$ is the speed of sound in air, and $\rho_0$ is the mean air density.

(i) Find the governing differential equation that must be satisfied by the pressure $p$ at all points within the volume $V$. [20%]

(ii) Find the governing differential equation for the membrane that must be satisfied by the displacement $u$ at all points on the surface $S$. [20%]

(iii) Show that the pressure and displacement must satisfy the following additional boundary condition at all points on $S$

$$\nabla p.\mathbf{n} = -\rho_0 \omega^2 u .$$ [20%]

(b) Prove the following vector identity

$$\mathbf{a}.\mathrm{curl}(\mathbf{b}\times\mathbf{x}) = 2\mathbf{a}.\mathbf{b} + \mathbf{a}.[(\mathbf{x}.\nabla)\mathbf{b}] - (\mathbf{a}.\mathbf{x})\mathrm{div}\mathbf{b}$$ [40%]

where the vectors $\mathbf{a}$ and $\mathbf{b}$ are functions of the position vector $\mathbf{x}$.

**END OF PAPER**