ENGINEERING TRIPOS PART IIB
ELECTRICAL AND INFORMATION SCIENCES TRIPOS PART II

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Tuesday 22 April 2003 2.30 to 4

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Module 4M13

LINEAR ALGEBRA AND OPTIMIZATION

*Answer not more than* ***three*** *questions.*

*The questions may be taken from any section.*

*All questions carry the same number of marks.*

*The* ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

*Answers to Sections A and B should be tied together and handed in separately.*

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| You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator |

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SECTION A

1 (a) Define the four fundamental subspaces of a matrix A and find a basis for these subspaces for the case:

$$A = \begin{pmatrix} 1 & 3 & -1 & 2 \\ 2 & 8 & -6 & -2 \\ 1 & -1 & 7 & -6 \end{pmatrix}$$

[40%]

(b) What is meant by the LU decompositions of a matrix and why is it useful? [10%]

(c) Find the LU decomposition of the matrix A above. [40%]

(d) What is the rank of A? [10%]

2 (a) What is meant by the Gram-Schmidt orthogonalisation? [15%]

Let $S$ be the space of solutions to the equation:

$$x_1 - x_2 + x_3 - 2x_4 = 0$$

(b) Show that the basis for the space $S$ is given by: [25%]

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(c) Use the Gram-Schmidt orthogonalisation process and the vectors in (b) to find an orthogonal basis for the space $S$. [40%]

(d) Use the basis found in (c) to find a vector in $S$ that is closest to the vector $b = (1, 1, 1, 1)^T$. [20%]

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SECTION B

3 (a) Describe briefly the differences between the *penalty function* and *barrier function* methods of constrained optimization, and discuss the advantages and disadvantages of each. [25%]

Shipping cartons are sometimes made with top and bottom formed by folding down flaps extending from each side as shown in Fig. 1.

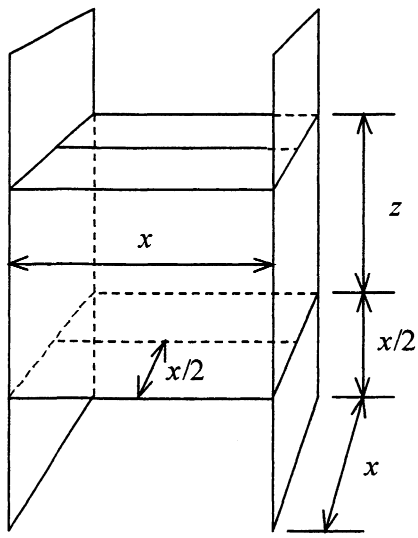


Fig. 1

Assume that a carton enclosing $V$ m$^3$ is required and that the thickness of the carton material $t << x, z$. In order to optimize the design of such cartons the amount of material used in their construction should be minimized.

(b) Explain why

$$f = 4xz + 4x^2 + p(x^2 z - V)^2$$

(in which $p$ is a penalty parameter) is a suitable objective function to be minimized successively (for increasing values of $p$) in order to find the solution to this optimization problem. [10%]

(cont.

(c) Perform three iterations of Newton's Method on this function for the case where $V = 2\ \text{m}^3$ and $p = 5$, starting from the initial solution $(x, z) = (1,1)$.

It can be easily shown (you are not required to do this) that the optimal shipping carton design enclosing a volume $V = 2\ \text{m}^3$ is $(x, z) = (1, 2)$. In view of this, comment on the performance of Newton's Method. [50%]

(d) What difficulties can occur when using Newton's Method? How can Newton's Method be modified to overcome these? [15%]

4 A cylindrical storage tank, of diameter $D$ and height $H$, which is closed at both ends, is to be fabricated to have a volume of $108\pi\ \text{m}^3$. The fabrication cost is proportional to the surface area of the sheet metal needed at a rate of £400 $\text{m}^{-2}$. The tank is to be housed in a shed with a sloping roof, which limits the height of the tank by the relation $H \le 12 - (D/2)$.

(a) Assuming that the thickness of the sheet metal is negligible compared to $D$ and $H$, show that a suitable objective function to be minimised is

$$f = 400\left(\frac{1}{2}\pi D^2 + \pi DH\right)$$

and identify the two constraint equations (one equality and one inequality) that must be satisfied. Bounds on the control variables ($D$ and $H$) can be ignored. [10%]

(b) By using suitable Lagrange and Kuhn-Tucker multipliers identify the four equations that give the first-order conditions that must be satisfied at an optimum. Do not use the equality constraint to eliminate a control variable. [10%]

(c) Given that both constraints are active when $D = 7.164$ m or $D = 22.256$ m, show that neither of these cases represents a minimum of $f$. [25%]

(d) Show that, if the inequality constraint is not active, a potential minimum of $f$ does exist at $D = 7.560$ m, and, by considering the second-order conditions, show that this is indeed a minimum. [45%]

(e) Estimate the change in the cost of the tank if the volume requirement is increased to $110\pi\ \text{m}^3$. [10%]

**END OF PAPER**